Solution to Exercise 6.1: We consider the following robust constraint

$$g(x,z) \leq t\,,\, \forall\, z \in \mathcal{D}(\rho)\ ,$$

where $g(x,z) := \mathrm{CVaR}_\alpha(r^T x; z)$.

First, we will work on the support function. We let $\mathcal{Z} := \mathcal{Z}_1 \cap \mathcal{Z}_2$ with $\mathcal{Z}_1$ the set discussed in theorem 6.6 and Z_2 be the KL-divergence set presented in table 6.1. Hence,

$$\begin{aligned}
\delta^*(v|\mathcal{Z}_1) &:= \min_{\lambda\in\mathbb{R}:\lambda\geq v} \lambda \\
\delta^*(v|\mathcal{Z}_2) &:= \min_{\mu\in\mathbb{R}:\mu\geq 0} \sum_{k=1}^{K} \frac{1}{K}\mu \exp(v_k/\mu - 1) + \rho\mu\ .
\end{aligned}$$

Based on the rule presented in table 6.1 for intersection of sets, this indicates us that support function for $\mathcal{Z}$ should be

$$\begin{aligned}
\delta^*(v|\mathcal{Z}) &= \min_{\lambda,\mu\geq 0, w^1, w^2:\lambda\geq w^1,\, w^1+w^2=v} \quad \lambda + \sum_{k=1}^{K} \frac{1}{K}\mu \exp(w_k^2/\mu - 1) + \rho\mu \\
&= \min_{\lambda,\mu\geq 0, w:\lambda\geq v-w} \quad \lambda + \sum_{k=1}^{K} \frac{1}{K}\mu \exp(w_k/\mu - 1) + \rho\mu\ .
\end{aligned}$$

Now, looking into $g_*(x,v)$, we start by laying out the detailed definition of this conjugate function:

$$\begin{aligned}
g_*(x,v) &:= \inf_{p\in\mathbb{R}^K:p\geq 0,\, \sum_k p_k=1} v^T p - \inf_s s + (1/\alpha)\sum_k p_k \max(-\bar{r}_k^T x - s;\, 0) \\
&= \inf_{p\in\mathbb{R}^K:p\geq 0,\, \sum_k p_k=1} \sup_s v^T p - s - (1/\alpha)\sum_k p_k \max(-\bar{r}_k^T x - s;\, 0) \\
&= \sup_s \inf_{p\in\mathbb{R}^K:p\geq 0,\, \sum_k p_k=1} v^T p - s - (1/\alpha)\sum_k p_k \max(-\bar{r}_k^T x - s;\, 0) \\
&= \sup_s \min_{k=1,\ldots,K} v_k - s - (1/\alpha)\max(-\bar{r}_k^T x - s;\, 0)\ ,
\end{aligned}$$

where we exploited Sion's minimax theorem exploiting the fact that the feasible set for p is bounded, and where we realized that a search over the worst-case distribution is simply a search over the worst-case outcome.

In conclusion, we can state that the robust CVaR optimization model takes the form:

$$
\begin{array}{ll}
\underset{x,s,t,\lambda,\mu,v,w}{\text{minimize}} & t \\
& \lambda + \sum_k \frac{1}{K}\mu \exp(w_k/\mu - 1) + \rho\mu - v_k + s + (1/\alpha)\max(-\bar{r}_k^T x - s;\, 0) \leq t\,,\ \forall\, k \\
& \lambda \geq v - w \\
& \mu \geq 0 \\
& \sum_i x_i = 1 \\
& x \geq 0\,,
\end{array}
$$

where $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$, $v \in \mathbb{R}^K$, $w \in \mathbb{R}^K$, $s \in \mathbb{R}$, and $t \in \mathbb{R}$.

Solution to Exercise 6.2: **Question 1:** The optimization problem can be reformulated as

$$
\begin{array}{ll}
\underset{x,y,t}{\text{maximize}} & t \\
\text{subject to} & t \leq \sum_i c_i y_i^{a_i} - c_i\,,\ \forall\, a \in \mathcal{U} \\
& y_i = 1 + x_i/d_i \\
& \sum_i p_i x_i \leq B \\
& x \geq 0\,,
\end{array}
$$

We therefore wish to use theorem 6.2 in order to obtain a tractable form for :

$$
t - \sum_i c_i y_i^{a_i} + c_i \leq 0\,,\ \forall\, a \in \mathcal{U}
$$

Considering first the function $g(y_i, a_i) := -\sum_i c_i y_i^{a_i}$, table 6.2 already tells us how to obtain the robust counterpart of $g'(y_i, a_i) := -y_i^{a_i}$, namely

$$
g'_*(y_i, v_i) = \frac{v_i}{\ln(y_i)} \ln\left(\frac{-v_i}{\ln(y_i)}\right) - \frac{v_i}{\ln(y_i)}\,,
$$

with the restriction that $v_i \leq 0$ otherwise the function evaluates to $-\infty$.

Using the help of theorem 6.9, we obtain that if $g''(y_i, a_i) := -c_i y_i^{a_i}$ then the conjugate function is

$$
g''_*(y_i, v_i) = c_i\left(\frac{v_i/c_i}{\ln(y_i)} \ln\left(\frac{-v_i/c_i}{\ln(y_i)}\right) - \frac{v_i/c_i}{\ln(y_i)}\right) = \frac{v_i}{\ln(y_i)} \ln\left(\frac{-v_i/c_i}{\ln(y_i)}\right) - \frac{v_i}{\ln(y_i)}\,.
$$

Finally, the table tells us how to handle sums of separable functions. Hence, we are left with the following robust counterpart:

$$t + \delta^*(v|\mathcal{U}) + \sum_i c_i - \sum_i \left(\frac{v_i}{\ln(y_i)} \ln \left(\frac{-v_i/c_i}{\ln(y_i)} \right) - \frac{v_i}{\ln(y_i)} \right) \leq 0$$
$$v \leq 0 \; .$$

Now, to obtain $\delta^*(v, |\mathcal{U}_1)$, we first note that $\mathcal{U}_1 = \bar{a} - 0.25\,\mathbf{diag}(\bar{a})\mathcal{Z}$, where $\mathbf{diag}(\bar{a}) = \sum_{i=1}^n \bar{a} e_i e_i^T$ (i.e. a matrix in $\mathbb{R}^{n\times n}$ with diagonal equal to $\bar{a}$) and with $\mathcal{Z}$ defined as the uncertainty set discussed in theorem 6.5. Hence,

$$\delta^*(v|\mathcal{Z}) := \min_{(\lambda,w)\in\mathbb{R}\times\mathbb{R}^m : \lambda \geq v - w\,,\, \lambda\geq 0,\, w\geq 0} \sum_i w_i + \Gamma\lambda \; .$$

Following theorem 6.7, we have that

$$\delta^*(v|\mathcal{U}) = \bar{a}^T v + \delta^*(-0.25\,\mathbf{diag}(\bar{a})v|\mathcal{Z}) \; ,$$

hence, that

$$\delta^*(v|\mathcal{U}) = \min_{(\lambda,w)\in\mathbb{R}\times\mathbb{R}^m : \lambda \geq -0.25\,\mathbf{diag}(\bar{a})v - w\,,\, \lambda\geq 0,\, w\geq 0} \quad \bar{a}^T v + \sum_i w_i + \Gamma\lambda \; .$$

Combining the two steps we get the tractable reformulation of the robust counterpart:

$$\begin{aligned}
\underset{x,y,t,\lambda,v,w}{\text{maximize}} \quad & t \\
\text{subject to} \quad & t + \bar{a}^T v + \sum_i w_i + \Gamma\lambda + \sum_i \left(c_i - \frac{v_i}{\ln(y_i)} \ln\left(\frac{-v_i/c_i}{\ln(y_i)} \right) + \frac{v_i}{\ln(y_i)} \right) \leq 0 \\
& \lambda \geq -0.25\,\mathbf{diag}(\bar{a})v - w \\
& v \leq 0 \\
& w \geq 0 \\
& \lambda \geq 0 \\
& y_i = 1 + x_i/d_i \\
& \sum_i p_i x_i \leq B \\
& x \geq 0 \; ,
\end{aligned}$$

Question 2: To obtain $\delta^*(v, |\mathcal{U}_2)$, we first note that $\mathcal{U}_2 = \bar{a} - 0.25\,\mathbf{diag}(\bar{a})\mathcal{Z}$, where $\mathbf{diag}(\bar{a}) = \sum_{i=1}^n \bar{a} e_i e_i^T$ (i.e. a matrix in $\mathbb{R}^{n\times n}$ with diagonal equal to $\bar{a}$) and with $\mathcal{Z}$ defined as

$$\mathcal{Z} := \{z \in \mathbb{R}^n \,|\, z \geq 0,\, \sum_i z_i = 1,\, \sum_i z_i \ln(z_i) \leq \rho\} \; .$$

To "speed up" the analysis, we observe that $\mathcal{Z} := \mathcal{Z}_1 \cap \mathcal{Z}_2$ with $\mathcal{Z}_1$ the set discussed in theorem 6.6 and Z_2 be the KL-divergence set presented in table 6.1. Hence,

$$\delta^*(v|\mathcal{Z}_1) := \min_{\lambda \in \mathbb{R}: \lambda \geq v} \lambda$$
$$\delta^*(v|\mathcal{Z}_2) := \min_{\mu \in \mathbb{R}: \mu \geq 0} \sum_i \mu \exp(v_i/\mu - 1) + \rho\mu \ .$$

Based on the rule presented in table 6.1 for intersection of sets, this indicates us that the support function for $\mathcal{Z}$ should be

$$\delta^*(v|\mathcal{Z}) = \min_{\lambda, \mu \geq 0, w^1, w^2 : \lambda \geq w^1,\, w^1 + w^2 = v} \quad \lambda + \sum_i \mu \exp(w_i^2/\mu - 1) + \rho\mu \ .$$

Following theorem 6.7, we have that

$$\delta^*(v|\mathcal{U}_2) = \bar{a}^T v + \delta^*(-0.25\,\mathbf{diag}(\bar{a})v|\mathcal{Z}) \ ,$$

hence, that

$$\begin{aligned}\delta^*(v|\mathcal{U}_2) &= \min_{\lambda, \mu \geq 0, w^1, w^2 : \lambda \geq w^1,\, w^1 + w^2 = 0.25\,\mathbf{diag}(\bar{a})v} \quad \bar{a}^T v + \lambda + \sum_i \mu \exp(w_i^1/\mu - 1) + \rho\mu \\ &= \min_{\lambda, \mu \geq 0, w : \lambda \geq 0.25\,\mathbf{diag}(\bar{a})v - w} \quad \bar{a}^T v + \lambda + \sum_i \mu \exp(w_i/\mu - 1) + \rho\mu \ .\end{aligned}$$

Combining the two steps we get the tractable reformulation of the robust counterpart:

$$\begin{aligned} \underset{x,y,t,s,\lambda,\mu,v,w}{\text{maximize}} \quad & t \\ \text{subject to} \quad & t + \bar{a}^T v + s + \sum_i \left(c_i - \frac{v_i}{\ln(y_i)} \ln\left(\frac{-v_i/c_i}{\ln(y_i)} \right) + \frac{v_i}{\ln(y_i)} \right) \leq 0 \\ & s \geq \lambda + \sum_i \mu \exp(w_i/\mu - 1) + \rho\mu \\ & \lambda \geq -0.25\,\mathbf{diag}(\bar{a})v - w \\ & v \leq 0 \\ & \mu \geq 0 \\ & y_i = 1 + x_i/d_i \\ & \sum_i p_i x_i \leq B \\ & x \geq 0 \ , \end{aligned}$$

Question 3: Refer to Matlab implementation using YALMIP in "Ex6_2.m".

Solution to Exercise 6.3: Consider the robust optimization problem:

$$\begin{array}{ll} \underset{x}{\text{maximize}} & \min_{z \in \mathcal{Z}} \sum_i x_i \exp(z_i) \\ \text{subject to} & \sum_i x_i \leq 1 \\ & x \geq 0 \ , \end{array}$$

where

$$\mathcal{Z} := \{z \in \mathbb{R}^n \,|\, \exists v \in [-1,1]^n, w \in [-1,1], z = \mu + Q(v+w), \|v\|_1 \leq \Gamma\} \ .$$

Question: Derive a tractable reformulation of this problem as a convex optimization problem of finite dimension ?

Solution: We can reformulate this problem as

$$\begin{array}{ll} \underset{x,t}{\text{maximize}} & t \\ \text{subject to} & -\sum_i x_i \exp(z_i) \leq t \ , \ z \in \mathcal{Z} \\ & \sum_i x_i \leq 1 \\ & x \geq 0 \ , \end{array}$$

We first look at the objective function $g(x,z) := -\sum_i x_i \exp(z_i)$. To find the partial concave conjugate, we start with $g'(x_i, z_i) := -x_i \exp(z_i)$. The partial concave conjugate of this function can be found as

$$g'_*(x_i, v_i) := \inf_{z_i} v_i z_i + x_i \exp(z_i) = v_i \ln(-v_i/x_i) - v_i \ ,$$

as long as $v_i \leq 0$ otherwise the infimum goes to $-\infty$. Based on the sum of separable functions rules, we get

$$g_*(x,v) := \sum_i v_i \ln(-v_i/x_i) - v_i \ .$$

Next, we need to identify the support function of $\mathcal{Z}$. Yet, in the description, we see that it is the affine mapping of the sum of two sets : $\mathcal{Z} := \mu + Q(\mathcal{Z}_1 + \mathcal{Z}_2)$, where $\mathcal{Z}_1$ is the budgeted uncertainty set, while $\mathcal{Z}_2 := 1 \cdot [-1,1]$, an affine projection of the

$[-1,1]$ interval. We therefore get:

$$\begin{aligned}
\delta^*(v|[-1,1]) &:= |v| \;\; \text{(based on support function of the box in } \mathbb{R}) \\
\delta^*(v|\mathcal{Z}_2) &:= |\sum_i v_i| \;\; \text{(based on theorem 6.7 and } \mathcal{Z}_2 := 1 \cdot [-1,1]) \\
\delta^*(v|\mathcal{Z}_1) &:= \min_{w^+ \geq 0, w^- \geq 0, \lambda \geq 0:\, \lambda \geq v - w^+,\, \lambda \geq -v - w^-} \sum_i w_i^+ + w_i^- + \Gamma\lambda \;\; \text{(based on corollary 6.8)} \\
\delta^*(v|\mathcal{Z}_1 + \mathcal{Z}_2) &:= \min_{w^+ \geq 0, w^- \geq 0, \lambda \geq 0: \lambda \geq v - w^+,\, \lambda \geq -v - w^-} \sum_i w_i^+ + w_i^- + \Gamma\lambda + |\sum_i v_i| \\
\delta^*(v|\mathcal{Z}) &:= \delta^*(v|\mu + Q(\mathcal{Z}_1 + \mathcal{Z}_2)) \\
&= \min_{w^+ \geq 0, w^- \geq 0, \lambda \geq 0: \lambda \geq Q^T v - w^+,\, \lambda \geq -Q^T v - w^-} \mu^T v + \sum_i w_i^+ + w_i^- + \Gamma\lambda + |\sum_i q_i^T v_i| \, ,
\end{aligned}$$

where q_i is the i-th column of Q. Note that the last two support function are obtained used the Minkowski sum rule from table 6.1 and theorem 6.7

Putting both of these analysis together we get:

$$\begin{aligned}
\underset{x,t,w^+,w^-,\lambda}{\text{maximize}} \quad & t \\
\text{subject to} \quad & \mu^T v + \sum_i w_i^+ + w_i^- + \Gamma\lambda + |\sum_i q_i^T v_i| - \sum_i (v_i \ln(-v_i/x_i) - v_i) \leq t \, , \; z \in \mathcal{Z} \\
& \lambda \geq Q^T v - w^+ \\
& \lambda \geq -Q^T v - w^- \\
& w^+ \geq 0, w^- \geq 0, \lambda \geq 0 \\
& \sum_i x_i \leq 1 \\
& x \geq 0 \, ,
\end{aligned}$$