Putting both of these analysis together we get:

$$\begin{aligned} \underset{x,t,v,w^+,w^-,\lambda}{\text{maximize}} \quad & t \\ \text{subject to} \quad & \mu^T v + \sum_i w_i^+ + w_i^- + \Gamma\lambda + |\sum_i q_i^T v_i| - (\sum_i (v_i \ln(-v_i/x_i) - v_i)) \leq -t\,, \\ & \lambda \geq Q^T v - w^+ \\ & \lambda \geq -Q^T v - w^- \\ & w^+ \geq 0, w^- \geq 0, \lambda \geq 0 \\ & \sum_i x_i \leq 1 \\ & x \geq 0\,, \end{aligned}$$

Solution to Exercise 8.1: Since $\mathcal{D}_1 = \mathcal{D}(\mathcal{Z}, \bar{\mu})$, we can exploit theorem 8.6 to reformulate this DRO has

$$\begin{aligned} \underset{x\in\mathcal{X},q,t,w^1,w^2}{\text{minimize}} \quad & t \\ \text{subject to} \quad & t \geq \delta^*(w^1|\mathcal{Z}) + \mu^T q - h_*^1(x, w^1 + q) \\ & t \geq \delta^*(w^2|\mathcal{Z}) + \mu^T q - h_*^2(x, w^2 + q)\,, \end{aligned}$$

where $h^1(x,z) := -\frac{1}{2}\xi^T Q(x)\xi$ and $h^2(x,z) := x^T C z$. We can first have a look at $\delta^*(v|\mathcal{Z})$ which is the support function of a polyhedron:

$$\delta^*(w|\mathcal{Z}) = \min_{\lambda\geq 0: W^T\lambda = w} v^T\lambda\,.$$

Next, we can obtain $h_*^2(x,w)$ using table 6.2 which tells us:

$$h_*^2(x,w) = \begin{cases} 0 & \text{if } w = C^T x \\ \infty & \text{otherwise} \end{cases}\,.$$

Finally, we can also exploit table 6.2 for $h_*^2(x,w)$ since $h^2(x,z) = -\sum_i z^T Q_i z x_i$:

$$h_*^1(x,w) = \sup_{s^1,s^2,\ldots,s^n:\sum_i s^i = w} -(1/2)\sum_i (1/x_i) s^i Q_i^{-1} s^i\,.$$

Assembling everything together, we get:

$$\begin{aligned} \underset{x\in\mathcal{X},q,t,w^1,w^2,\lambda^1,\lambda^2,s^1,\ldots,s^n}{\text{minimize}} \quad & t \\ \text{subject to} \quad & t \geq v^T\lambda^1 + \mu^T q + (1/2)\sum_i (1/x_i) s^i Q_i^{-1} s^i \\ & \lambda^1 \geq 0\,,\ W^T\lambda^1 = w^1 \\ & \sum_i s^i = w^1 + q \\ & t \geq v^T\lambda^2 + \mu^T q \\ & \lambda^2 \geq 0\,,\ W^T\lambda^2 = w^2 \\ & w^2 + q = C^T x\,. \end{aligned}$$

Solution to Exercise 8.2: We can easily remark that the DRO reduces to

$$\underset{x\in\mathcal{X}}{\text{minimize}} \quad \max_{\mu\in\mathcal{U}_1'(\Gamma)} \max_{F\in\mathcal{D}(\mathcal{Z},\mu)} \mathbb{E}_F[\max(-\frac{1}{2}\xi^T Q(x)\xi,\ x^T C\xi)]\,,$$

where

$$\mathcal{U}_1'(\Gamma) = \{\mu \,|\, \exists\Delta \in \mathcal{U}_1(\Gamma),\, \mu = \bar{\mu} + \Delta\}\,,$$

with

$$\mathcal{U}_1(\Gamma) := \{\Delta \in \mathbb{R}^m \,|\, \Delta \geq 0 \sum_{i=1}^{m} \Delta_i \leq \Gamma\}\,.$$

Based on corollary 8.9, one should understand that the DRO now becomes equivalent to :

$$\begin{aligned}
\underset{x\in\mathcal{X},q,t,w^1,w^2,\lambda^1,\lambda^2,s^1,\ldots,s^n}{\text{minimize}} \quad & t + \delta^*(q|\mathcal{U}_1'(\Gamma)) \\
\text{subject to} \quad & t \geq v^T\lambda^1 + (1/2)\sum_i (1/x_i) s^i Q_i^{-1} s^i \\
& \lambda^1 \geq 0\,,\ W^T\lambda^1 = w^1 \\
& \sum_i s^i = w^1 + q \\
& t \geq v^T\lambda^2 \\
& \lambda^2 \geq 0\,,\ W^T\lambda^2 = w^2 \\
& w^2 + q = C^T x\,.
\end{aligned}$$

where $\mathcal{U}_1'(\Gamma) = \{\mu \,|\, \exists\Delta \in \mathcal{U}_1(\Gamma),\, \mu = \bar{\mu} + \Delta\}$. Based on theorem 6.7, we have that $\delta^*(q|\mathcal{U}_1'(\Gamma)) = q^T\bar{\mu} + \delta^*(q|\mathcal{U}_1(\Gamma)$.

$$\delta^*(q|\mathcal{U}_1(\Gamma)) = \sup_{\Delta\in\mathcal{U}_1(\Gamma)} q^T\Delta = \min_{\omega\geq 0:\omega\geq q} \Gamma\omega\,.$$

Hence, we obtain:

$$\begin{aligned}
\underset{x\in\mathcal{X},q,t,w^1,w^2,\lambda^1,\lambda^2,s^1,\ldots,s^n,\omega}{\text{minimize}} \quad & t + \bar{\mu}^T q + \Gamma\omega \\
\text{subject to} \quad & \omega \geq 0\,,\ \omega \geq q \\
& t \geq v^T\lambda^1 + (1/2)\sum_i (1/x_i) s^i Q_i^{-1} s^i \\
& \lambda^1 \geq 0\,,\ W^T\lambda^1 = w^1 \\
& \sum_i s^i = w^1 + q \\
& t \geq v^T\lambda^2 \\
& \lambda^2 \geq 0\,,\ W^T\lambda^2 = w^2 \\
& w^2 + q = C^T x\,.
\end{aligned}$$

Solution to Exercise 8.3: The answer to this question is easier to reach if we realize that our previous efforts led us to:

$$\sup_{F\in\mathcal{D}_2(\Gamma)} \mathbb{E}_F[\max(-\frac{1}{2}\xi^T Q(x)\xi,\ x^T C\xi)] = \underset{(q,t,\omega)\in\mathcal{Q}}{\text{minimize}} \quad t+\bar{\mu}^T q+\Gamma\omega\,,$$

where

$$\mathcal{Q} := \left\{ (q,t,\omega) \;\middle|\; \exists w^1, w^2, \lambda^1, \lambda^2, s^1, \ldots, s^n, \omega\,,\ \begin{array}{l} \omega \geq 0\,,\ \omega \geq q \\ t \geq v^T\lambda^1 + (1/2)\sum_i (1/x_i) s^i Q_i^{-1} s^i \\ \lambda^1 \geq 0\,,\ W^T\lambda^1 = w^1 \\ \sum_i s^i = w^1 + q \\ t \geq v^T\lambda^2 \\ \lambda^2 \geq 0\,,\ W^T\lambda^2 = w^2 \\ w^2 + q = C^T x\,. \end{array} \right\}$$

Hence

$$\begin{aligned}
&\sup_{\Gamma\in[0,\bar{\Gamma}]} \sup_{F\in\mathcal{D}_2(\Gamma)} \mathbb{E}_F[\max(-\frac{1}{2}\xi^T Q(x)\xi,\ x^T C\xi)] - \alpha\Gamma \\
&\quad = \sup_{\Gamma\in[0,\bar{\Gamma}]} \min_{(q,t,\omega)\in\mathcal{Q}} \; f(t,q,\omega,\Gamma) := t+\bar{\mu}^T q + \Gamma(\omega-\alpha) \\
&\quad = \min_{(q,t,\omega)\in\mathcal{Q}} \; t+\bar{\mu}^T q + \sup_{\Gamma\in[0,\bar{\Gamma}]} \Gamma(\omega-\alpha) \\
&\quad = \min_{(q,t,\omega)\in\mathcal{Q}} \; t+\bar{\mu}^T q + \max(0,\ \bar{\Gamma}(\omega-\alpha))\,,
\end{aligned}$$

where we first applied Sion's minimax theorem since Γ is in a bounded set and the function $f(t,q,\omega,\Gamma)$ is linear in both (t,q,ω) and Γ. The last equality comes from the fact that since $\Gamma(\omega-\alpha)$ is linear in Γ, then the supremum over Γ necessarily either occurs at $\Gamma=0$ or $\Gamma=\bar{\Gamma}$.

We can finally reintegrate this expression inside the globalized distributionally robust optimization problem presented in this question:

$$
\begin{array}{rl}
\underset{x\in\mathcal{X},q,t_1,t_2,w^1,w^2,\lambda^1,\lambda^2,s^1,\ldots,s^n,\omega}{\text{minimize}} & t_1 \\
\text{subject to} & t_2 + \bar{\mu}^T q \leq t_1 \\
& t_2 + \bar{\mu}^T q + \Gamma(\omega - \alpha) \leq t_1 \\
& \omega \geq 0 \ , \ \omega \geq q \\
& t_2 \geq v^T \lambda^1 + (1/2) \sum_i (1/x_i) s^i Q_i^{-1} s^i \\
& \lambda^1 \geq 0 \ , \ W^T \lambda^1 = w^1 \\
& \sum_i s^i = w^1 + q \\
& t_2 \geq v^T \lambda^2 \\
& \lambda^2 \geq 0 \ , \ W^T \lambda^2 = w^2 \\
& w^2 + q = C^T x \, .
\end{array}
$$