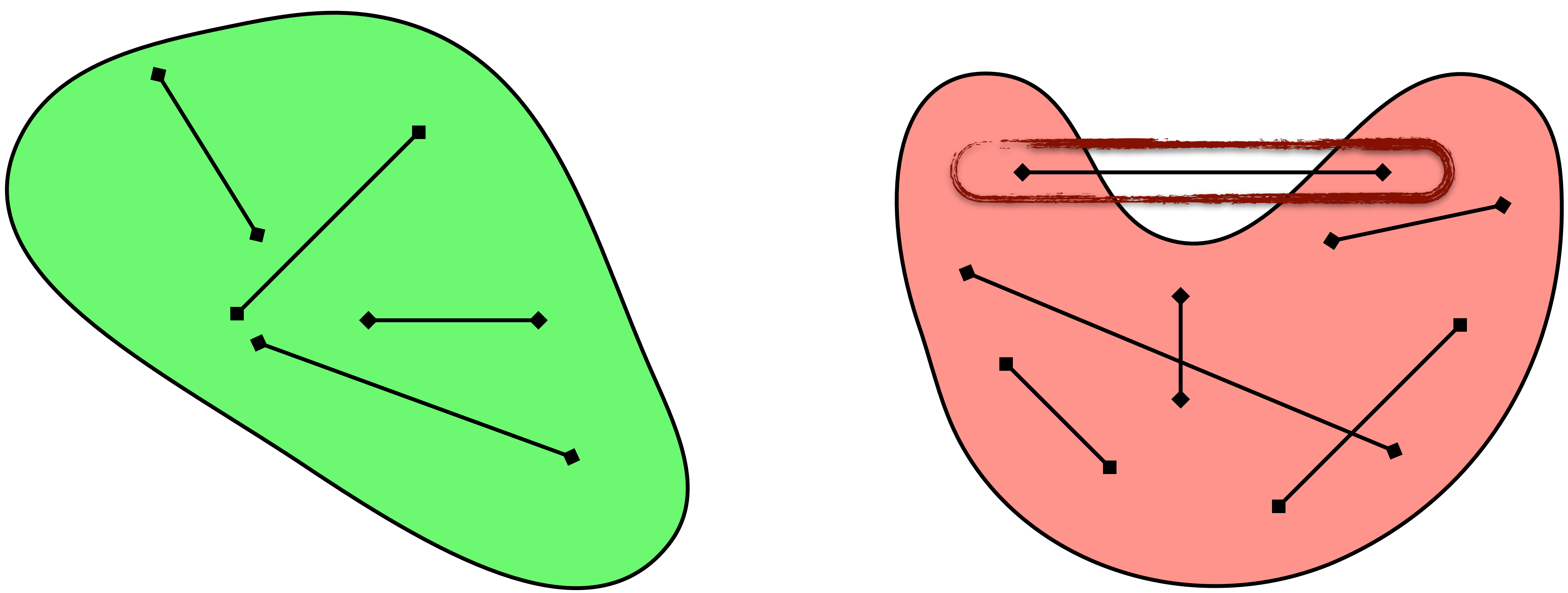
Preface: An Introduction to Convex Analysis

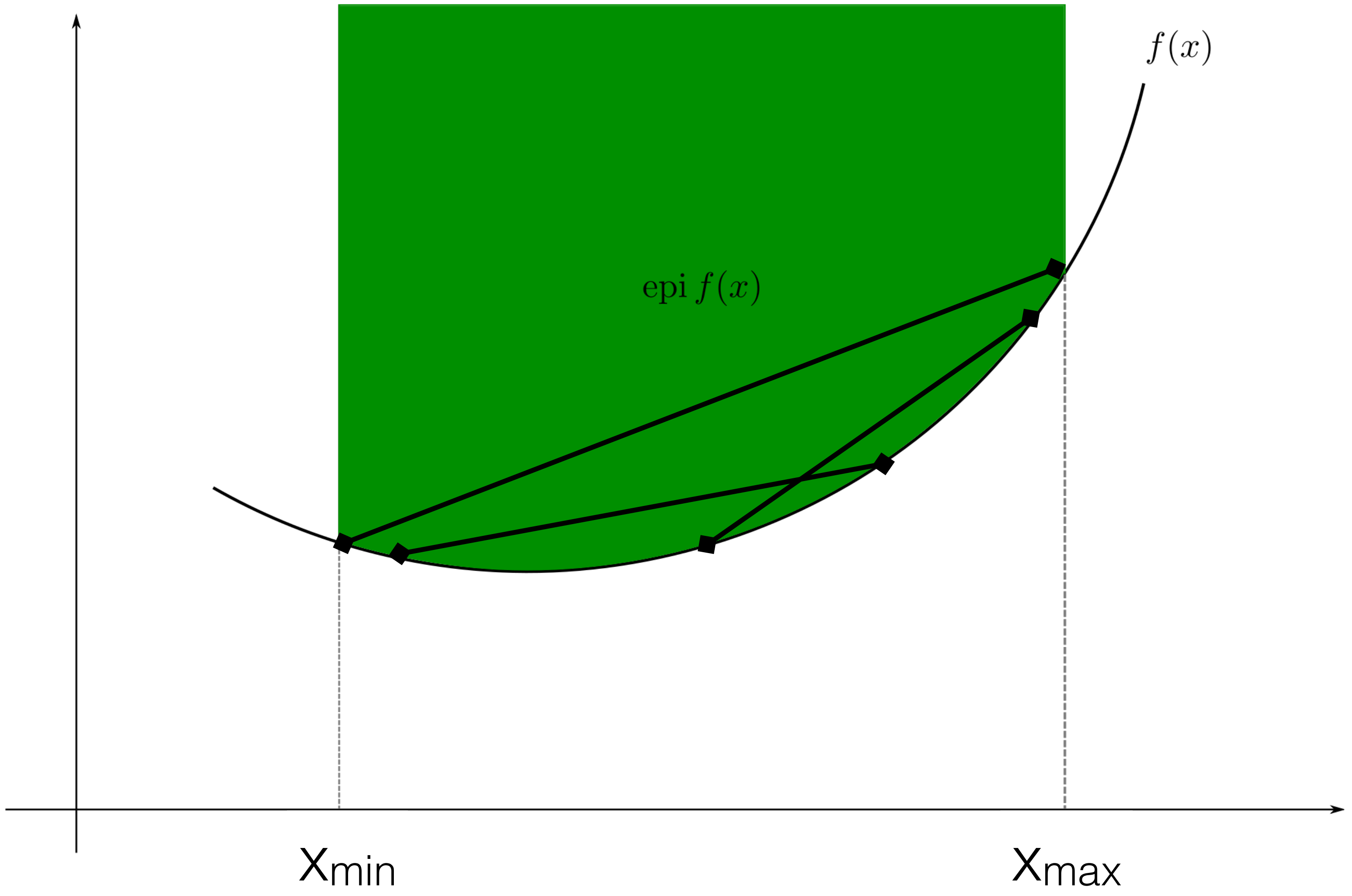
What is a convex set?

Definition 10.1 : A set $\mathcal{X} \subseteq \mathbb{R}^n$ is convex if for any two members $x_1 \in \mathcal{X}$ and $x_2 \in \mathcal{X}$, any "convex combination" of these two points is also a member of $\mathcal{X}$. Namely, for all $\theta \in [0, 1]$, we have that $\theta x_1 + (1 - \theta) x_2 \in \mathcal{X}$.



What is a convex function?

Definition 10.2 : A function $h : \mathcal{X} \to \mathbb{R}$, with $\mathcal{X} \subseteq \mathbb{R}^n$ as its domain, is said to be convex if its epigraph is a convex set. Namely, it is convex if and only if $\mathcal{X}$ is a convex set and that for any two members x_1 and x_2 of $\mathcal{X}$, and any convex combination $x_3 := \theta x_1 + (1-\theta)x_2$, with $\theta \in [0,1]$, of these two points, it is the case that $h(x_3) \leq \theta h(x_1) + (1-\theta)h(x_2)$.



What is a concave function?

Definition 10.3 : A function $h : \mathcal{X} \to \mathbb{R}$, with $\mathcal{X} \subseteq \mathbb{R}^n$ as its domain, is said to be concave if the function $-h(x)$ is convex. Namely, it is concave if $\mathcal{X}$ is a convex set, and if for any two members x_1 and x_2 of $\mathcal{X}$, and any convex combination $x_3 := \theta x_1 + (1-\theta)x_2$, with $\theta \in [0,1]$, of these two points, it is the case that $h(x_3) \geq \theta h(x_1) + (1-\theta)h(x_2)$.

- In other words, a function f(x) is concave if its negative is convex.

$$-f(x) \text{ convex } \Leftrightarrow f(x) \text{ concave}$$

Operations that preserves convexity

- Addition of two convex function. Namely, if $g_1(x)$ and $g_2(x)$ are convex functions then $g_1(x) + g_2(x)$ is a convex function.

- Multiplying a convex function by a positive scalar. Namely, if $g(x)$ is convex then $\alpha g(x)$ is convex for any $\alpha \geq 0$.

- Taking the supremum over a set of convex functions. Namely, if $g(x, z)$ is convex for all $z \in \mathcal{Z}$ then $\sup_{z \in \mathcal{Z}} g(x, z)$ is convex

- Taking the infimum over a subset of variables for which the function is jointly convex. Namely, if $g(x, y)$ is jointly convex in x and y, and $\mathcal{X}$ is convex and non-empty, then $\inf_{x \in \mathcal{X}} g(x, y)$ is convex in y.

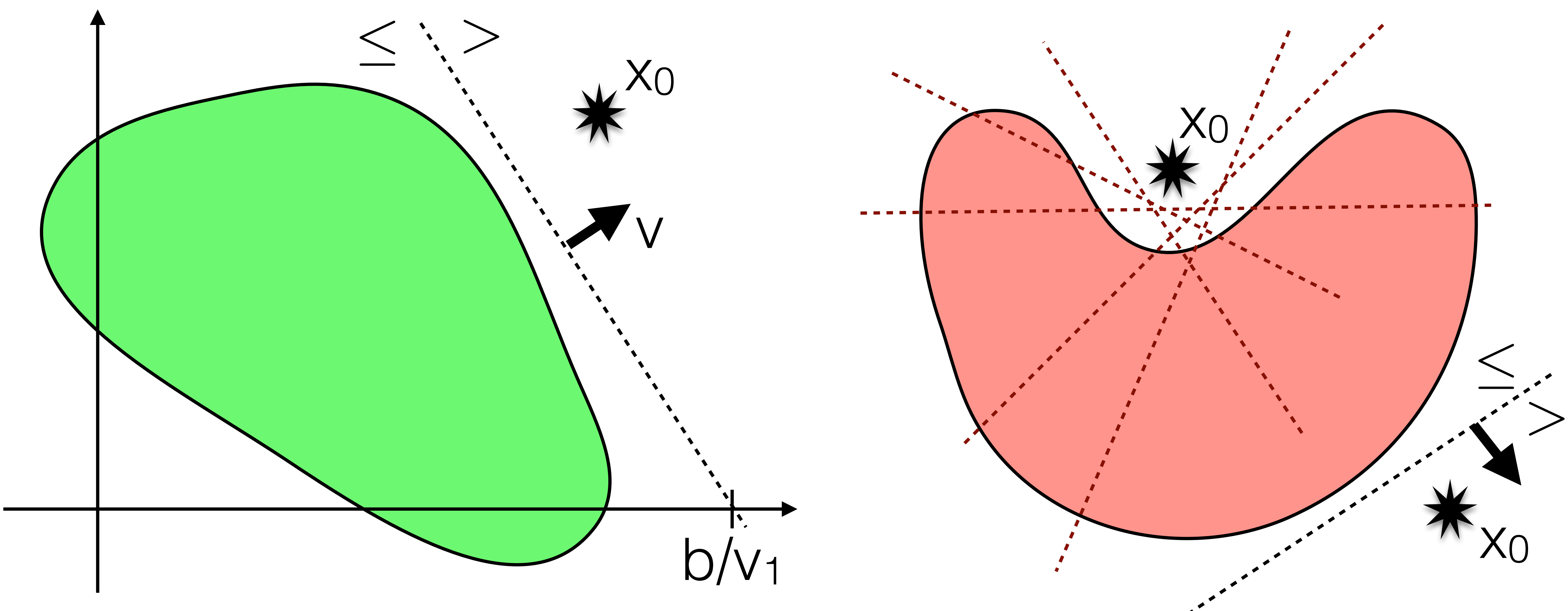
More operations that preserves convexity

- Any composition of a convex function with an affine mapping. Namely, if $g : \mathbb{R}^n \to \mathbb{R}$ while $A \in \mathbb{R}^{n\times m}$ and $b \in \mathbb{R}^n$,then $g(Ax + b)$ is convex in x
- Some composition of convex and monotone functions. Namely, let $h : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ then $h(g(x))$ is convex in x if one of the conditions below apply:
  - $h(\cdot)$ is convex and nondecreasing and $g(\cdot)$ is convex
  - $h(\cdot)$ is convex and nonincreasing and $g(\cdot)$ is concave
- The perspective of a convex function. Namely, if $g(x)$ is convex, then $tg(x/t)$ is jointly convex in t and x as long as $t > 0$.

Strict separating hyperplane theorem & Farkas lemma

Theorem 10.4 :*(Strict separating hyperplane theorem) Let $\mathcal{X} \in \mathbb{R}^n$ be a closed convex set and $x_0 \notin \mathcal{X}$. Then there exists a hyperplane parametrized by $v \in \mathbb{R}^n$ and $b \in \mathbb{R}$ that strictly separates x_0 from $\mathcal{X}$. Namely,*

$$v^T x \leq b, \forall x \in \mathcal{X} \qquad \& \qquad v^T x_0 > b$$



Strict separating hyperplane theorem & Farkas lemma

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Farkas lemma:

Lemma 2.4. : *Let W be a real $s \times m$ matrix and x be an m-dimensional vector. Then, exactly one of the following two statements is true:*

1. *There exists a $\lambda \in \mathbb{R}^s$ such that $W^T \lambda = x$ and $\lambda \geq 0$.*
2. *There exists a $\Delta \in \mathbb{R}^m$ such that $W\Delta \leq 0$ and $x^T \Delta > 0$.*

A polyhedron is non-empty (statement 1)

A certificate of emptiness exists (statement 2)