Solutions to Exercises

Solution to Exercise 2.1: One might consider that the constraint in this expression is equivalent to

$$(\sum_i \theta_i \bar{z}_i)^T x \leq b - a^T x \,, \forall \theta \in \mathcal{U} \,,$$

where $\mathcal{U} := \{\theta \in \mathbb{R}^K \,|\, \theta \geq 0, \sum_i \theta_i = 1\}$. This constraint is further equivalent to

$$(\mathbb{Z}\theta)^T x \leq b - a^T x \,, \forall \theta \in \mathcal{U} \,,$$

where $\mathbb{Z} := [\ \bar{z}_1 \quad \dots \quad \bar{z}_K\ ]$. this is a form where theorem 2.7 can be applied, considering that $J = 1$, $p_0 := c$, $P_1 := \mathbb{Z}$, $r_1 = -1$, and W and v are as follow:

$$W := \begin{bmatrix} \mathbf{1}_K^T \\ -\mathbf{1}_K^T \\ -I \end{bmatrix} \qquad\qquad v := \begin{bmatrix} 1 \\ -1 \\ \mathbf{0}_K \end{bmatrix} .$$

This gives us the following LP

$$\begin{array}{ll} \underset{x,\mu_1,\mu_2,\lambda}{\text{maximize}} & c^T x \\ \text{subject to} & \mu_1 - \mu_2 \leq b - a^T x \\ & \mu_1 - \mu_2 - \lambda = \mathbb{Z}^T x \\ & \mu_1 \geq 0, \mu_2 \geq 0, \lambda \geq 0 \\ & 0 \leq x \leq 1 \\ & \displaystyle\sum_{i=1}^{n} x_i \leq 1 \,, \end{array}$$

where $\mu_1 \in \mathbb{R}$, $\mu_2 \in \mathbb{R}$, and $\lambda \in \mathbb{R}^K$.

We can then replace the expression $\mu_1 - \mu_2$ with $\mu \in \mathbb{R}$ which leaves us with

$$\begin{array}{ll} \underset{x,\mu,\lambda}{\text{maximize}} & c^T x \\ \text{subject to} & \mu \leq b - a^T x \\ & \mu - \lambda = \mathbb{Z}^T x \\ & \lambda \geq 0 \\ & 0 \leq x \leq 1 \\ & \displaystyle\sum_{i=1}^{n} x_i \leq 1 \ , \end{array}$$

Then, since λ is only involved in one equality constraint other than the non-negativity one, we can replace that equality constraint with

$$\mu \geq \mathbb{Z}^T x \ .$$

This leaves us with the final option of replacing the μ with the largest amount it can take which is $b - a^T x$. We get the constraint

$$b - a^T x \geq \mathbb{Z}^T x \ ,$$

which is equivalent to

$$\bar{z}_i^T x \leq b - a^T x \, , \forall\, i = 1, \ldots, K \ .$$

In conclusion, the reformulation reduces to

$$\begin{array}{ll} \underset{x}{\text{maximize}} & c^T x \\ \text{subject to} & (a + \bar{z}_i)^T x \leq b \, , \forall\, i = 1, \ldots, K \\ & 0 \leq x \leq 1 \\ & \displaystyle\sum_{i=1}^{n} x_i \leq 1 \ , \end{array}$$

which is exactly saying that x must satisfy the constraint for each scenarios $\bar{z}_i$. Recall that this is what we did in example 2.1. We actually just proved a version of the well known principle of robust optimization which states that the robust constraint

$$g(x, z) \leq 0 \, , \forall\, z \in \mathcal{Z}$$

is equivalent to the robust constraint

$$g(x, z) \leq 0 \, , \forall\, z \in \text{ConvexHull}(\mathcal{Z}) \ ,$$

when $g(x, z)$ is affine with respect to z.

Solution to Exercise 2.2: In order to employ theorem 2.7, we need to describe the uncertainty set in the form $Wz \leq v$ which is not currently the case. Our first step will therefore be to raise the uncertainty space in $\mathbb{R}^{2m}$ as follow

$$\mathcal{Z}'(\Gamma) := \left\{ z' \in \mathbb{R}^{2m} \,\middle|\, \exists\, z \in \mathbb{R}^m,\, s \in \mathbb{R}^m,\, z' = [\, z^T \quad s^T \,]^T,\, -s \leq z \leq s,\, s \leq 1,\, \sum_i s_i \leq \Gamma \right\} .$$

In this uncertainty space, the robust constraint is equivalent to

$$(a + [\, \boldsymbol{I} \quad 0 \,] z')^T x \leq b \,,\, \forall\, z' \in \mathcal{Z}'(\Gamma) \,.$$

We can therefore consider that $J = 1$, $p_1 = a$, $P_1 = [\, \boldsymbol{I} \quad 0 \,]$, $q_1 = 0$, $r_1 = b$, and that

$$W := \begin{bmatrix} -I & -I \\ I & -I \\ 0 & I \\ 0 & \mathbf{1}_m \end{bmatrix} \qquad\qquad v := \begin{bmatrix} \mathbf{0}_m \\ \mathbf{0}_m \\ \mathbf{1}_m \\ \Gamma \end{bmatrix}$$

Hence, the reduced robust counterpart takes the following form

$$\begin{array}{ll} \underset{x,\lambda_1,\lambda_2,\lambda_3,\lambda_4}{\text{maximize}} & c^T x \\ \text{subject to} & a^T x + \mathbf{1}_m^T \lambda_3 + \Gamma \lambda_4 \leq b \\ & x = \lambda_2 - \lambda_1 \\ & \lambda_3 + \lambda_4 = \lambda_1 + \lambda_2 \\ & \lambda_1 \geq 0 \,, \lambda_2 \geq 0 \,, \lambda_3 \geq 0 \,, \lambda_4 \geq 0 \\ & 0 \leq x \leq 1 \\ & \displaystyle\sum_{i=1}^n x_i \leq 1 \,, \end{array}$$

where $\lambda_1 \in \mathbb{R}^m$, $\lambda_2 \in \mathbb{R}^m$, $\lambda_3 \in \mathbb{R}^m$, and $\lambda_4 \in \mathbb{R}$.

Solution to Exercise 2.3: We employ a similar approach as in exercise 2.1, meaning that we first reformulate in terms of θ being the uncertain vector. This leads to the following robust counterpart:

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & (a + \mathbb{Z}\theta)^T x \leq b \,,\, \forall\, \theta \in \mathcal{U} \\ & 0 \leq x \leq 1 \\ & \displaystyle\sum_{i=1}^n x_i \leq 1 \,, \end{array}$$

where $\mathcal{U} := \{\theta \in \mathbb{R}^K \,|\, \theta \geq 0, \sum_{i=1}^K \theta_i = 1, \theta \leq \frac{1}{K\alpha}\}$. To apply theorem 2.7, we need to characterize the different elements of the LP-RC. Specifically, we say that $q = a$, $P = \mathbb{Z}$, $r = b$, $p = 0$, and finally that

$$W := \begin{bmatrix} -I \\ I \\ \mathbf{1}_K^T \\ -\mathbf{1}_K^T \end{bmatrix} \qquad \& \qquad v := \begin{bmatrix} \mathbf{0}_K \\ \frac{1}{K\alpha}\mathbf{1}_K \\ 1 \\ -1 \end{bmatrix} .$$

This leads to the following LP reformulation

$$\begin{aligned} \underset{x,\lambda_1,\lambda_2,\lambda_3,\lambda_4}{\text{maximize}} \quad & c^T x \\ \text{subject to} \quad & a^T x + \frac{1}{K\alpha}\mathbf{1}_K^T \lambda_2 + \lambda_3 - \lambda_4 \leq b \\ & \mathbb{Z}^T x = -\lambda_1 + \lambda_2 + \mathbf{1}_K \lambda_3 - \mathbf{1}_K \lambda_4 \\ & \lambda_1 \geq 0 \,, \lambda_2 \geq 0 \,, \lambda_3 \geq 0 \,, \lambda_4 \geq 0 \\ & 0 \leq x \leq 1 \\ & \sum_{i=1}^n x_i \leq 1 \,, \end{aligned}$$

where $\lambda_1 \in \mathbb{R}^K$, $\lambda_2 \in \mathbb{R}^K$, $\lambda_3 \in \mathbb{R}$, and $\lambda_4 \in \mathbb{R}$. Given that λ_3 and λ_4 always appear in the expression $\lambda_3 - \lambda_4$, we can simply replace the expression with $s := \lambda_3 - \lambda_4$. Also, given that λ_1 is only involved in one constraint, it can be removed from the problem after replacing the equality constraint with an inequality in the appropriate direction. Overall, we get the following reduced LP:

$$\begin{aligned} \underset{x,\lambda_2,s}{\text{maximize}} \quad & c^T x \\ \text{subject to} \quad & a^T x + \frac{1}{K\alpha}\mathbf{1}_K^T \lambda_2 + s \leq b \\ & \mathbb{Z}^T x \leq \lambda_2 + \mathbf{1}_K s \\ & \lambda_2 \geq 0 \\ & 0 \leq x \leq 1 \\ & \sum_{i=1}^n x_i \leq 1 \,, \end{aligned}$$