Chapter 6:

Robust Nonlinear Programming

Our robust nonlinear programming formulation

- We focus on the following formulation:

$$g(x,z) \leq 0\,,\; \forall\, z \in \mathcal{Z}$$

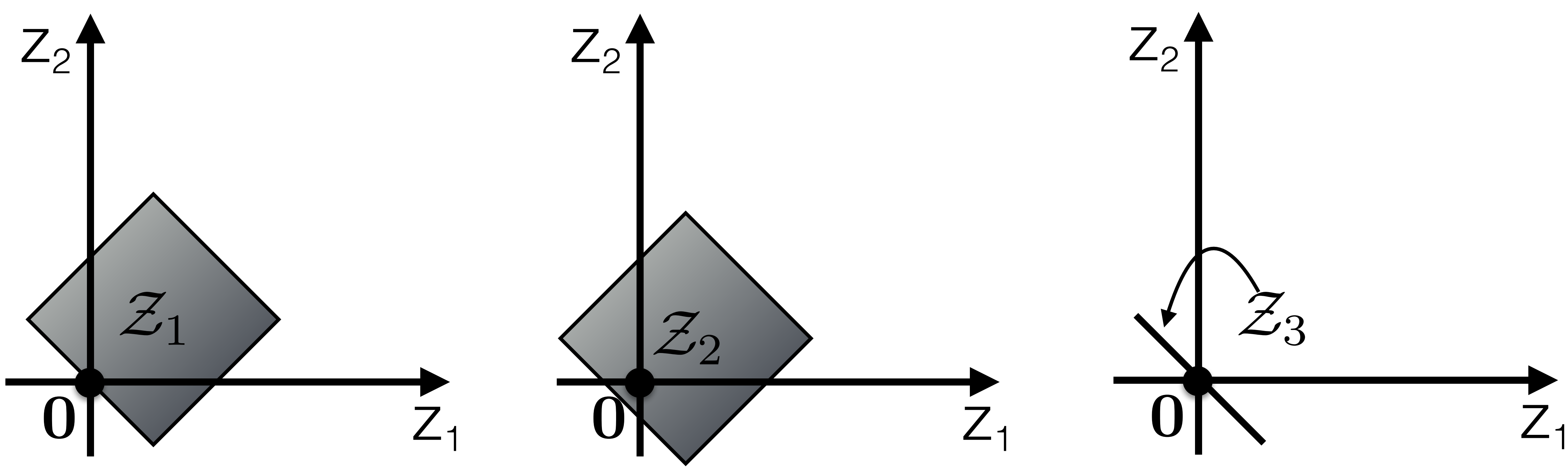
where:

- $g(\cdot,\cdot)$ is a mapping defined over the convex domain $\mathcal{X}_g \times \mathcal{Z}_g$
- g(x,z) is convex in x and concave in z
- $\mathcal{Z} \subset \mathbb{R}^m$ is a given non-empty, convex, and compact set
- there exist a z_0 in the relative interior of $\mathcal{Z}$ intersect $\mathcal{Z}_g$

Relative interior definition

- There exists a ball centered at z_0 and radius $\epsilon > 0$ which projection on the affine space spanned by $\mathcal{Z}$ is included in $\mathcal{Z}$
  For convex uncertainty sets this translates:

$$\exists \epsilon > 0, \forall z \in \mathcal{Z}, z_0 - \epsilon((z - z_0)/\|z - z_0\|_2) \in \mathcal{Z}\}$$



$\mathbf{0} \notin \text{relint}(\mathcal{Z}_1)$ $\mathbf{0} \in \text{relint}(\mathcal{Z}_2)$ $\mathbf{0} \in \text{relint}(\mathcal{Z}_3)$

Fenchel Robust Counterpart

Assumptions:

- g(x,z) is convex in x and concave in z
- $\mathcal{Z} \subset \mathbb{R}^m$ is a given non-empty, convex, and compact set
- there exist a z_0 in the relative interior of both $\mathcal{Z}$ and $\mathcal{Z}_g$

Theorem 6.2. : *The vector $x \in \mathcal{X}$ satisfies the robust constraint* (6.1) *if and only if $x \in \mathcal{X}$ and $v \in \mathbb{R}^m$ satisfy the single inequality*

$$(FRC) \quad \delta^*(v|\mathcal{Z}) - g_*(x, v) \leq 0 \; , \tag{6.2}$$

where the support function δ^* is defined as

$$\delta^*(v|\mathcal{Z}) := \sup_{z \in \mathcal{Z}} z^T v$$

and the partial concave conjugate function g_* is defined as

$$g_*(x, v) := \inf_{z \in \mathcal{Z}_g} v^T z - g(x, z) \; .$$

When $\mathcal{Z}$ is bounded strong duality follows from Sion's minimax theorem

Lemma 4.4. :*(Sion's minimax theorem [32]) Let $\mathcal{X} \subset \mathbb{R}^n$ be a convex set and $\mathcal{Z} \in \mathbb{R}^m$ be a compact convex set, and let h be a real-valued function on $\mathcal{X} \times \mathcal{Z}$ with*

1. *$h(x, \cdot)$ lower semicontinuous and quasi-convex on $\mathcal{Z}$, $\forall\, x \in \mathcal{X}$*
2. *$h(\cdot, z)$ upper semicontinuous and quasiconcave on $\mathcal{X}$, $\forall\, z \in \mathcal{Z}$*

then

$$\sup_{x \in \mathcal{X}} \min_{z \in \mathcal{Z}} \; h(x, z) = \min_{z \in \mathcal{Z}} \sup_{x \in \mathcal{X}} \; h(x, z) \; .$$

- Coro #1: If both $\mathcal{X}$ and $\mathcal{Z}$ are convex, one of them is bounded, and h(x,z) is concave in x & convex in z, then

$$\sup_{x \in \mathcal{X}} \inf_{z \in \mathcal{Z}} h(x, z) \quad = \quad \inf_{z \in \mathcal{Z}} \sup_{x \in \mathcal{X}} h(x, z)$$

- Coro #2: If both $\mathcal{X}$ and $\mathcal{Z}$ are convex, one of them is bounded, and h(x,z) is convex in x & concave in z, then

$$\inf_{x \in \mathcal{X}} \sup_{z \in \mathcal{Z}} h(x, z) \quad = \quad \sup_{z \in \mathcal{Z}} \inf_{x \in \mathcal{X}} h(x, z)$$

Example : Quadratic Programming

Example 6.3. : Consider the following robust optimization constraint:

$$p(x)^T z + s(x) - z^T P(x) z \leq 0\,,\ \forall\, z \in \mathcal{Z}\ ,$$

where $p : \mathbb{R}^n \to \mathbb{R}^m$ is an affine function of x, $s : \mathbb{R}^n \to \mathbb{R}$, and $P(x) : \mathbb{R}^n \to \mathbb{R}^{m\times m}$, and finally where

$$\mathcal{Z} := \{z \in \mathbb{R}^m \,|\, z^T Q z \leq r\}\ ,$$

with $Q \in \mathbb{R}^{m\times m}$ a symmetric matrix and $r \in \mathbb{R}$.

After describing $g(x,z)$ as $g(x,z) := p(x)^T z + s(x) - z^T P(x) z$ and letting $z_0 = 0$, one needs to make the following assumptions in order to apply theorem 6.2:

- Impose that $x \in \mathcal{X}_g$ with $\mathcal{X}_g := \{x \,|\, P(x) \succeq 0\}$, namely that we have the guarantee that $P(x)$ is positive semi-definite in order to make $g(x,z)$ concave in z.
- Impose that $Q \succ 0$ and that $r > 0$, namely that Q is positive definite to ensure that $\mathcal{Z}$ is convex and bounded, and that $0 \in \mathrm{relint}(\mathcal{Z})$.

When applying theorem 6.2, we obtain that the constraint is equivalent to

$$\exists v \in \mathbb{R}^m\ \ \delta^*(v|\mathcal{Z}) - g_*(x,v) \leq 0\,,$$

Example : Quadratic Programming under Polyhedron

- It's really rather simple to obtain the robust counterpart under a different uncertainty set. Simply replace the conjugate of the support function.

- Example: using Bz≤b instead of ellipsoid

Ellipsoidal set

$$\delta^*(v|\mathcal{Z}) = \sqrt{r}\|Q^{-1/2}v\|_2 \longrightarrow \begin{array}{l} \sqrt{r}\|Q^{-1/2}v\|_2 - t \leq 0 \\ \begin{bmatrix} P(x) & (v-p(x))/2 \\ (v-p(x))^T/2 & -s(z)-t \end{bmatrix} \succeq 0 \end{array}$$

Example : Quadratic Programming under Polyhedron

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Polyhedral set

$$\delta^*(v|\mathcal{Z}) = \inf_{\lambda:\lambda\geq 0, B^T\lambda = v} b^T\lambda \quad \longrightarrow \quad \begin{aligned} & b^T\lambda - t \leq 0 \\ & B^T\lambda = v \\ & \lambda \geq 0 \\ & \begin{bmatrix} P(x) & (v-p(x))/2 \\ (v-p(x))^T/2 & -s(z)-t \end{bmatrix} \succeq 0 \end{aligned}$$

Some tractable reformulation are beyond the reach of FRC

Example 6.4. : Consider the following robust optimization constraint:

$$p(x)^T z + s(x) - z^T P(x) z \leq 0\,, \forall\, z \in \mathcal{Z}\ ,$$

where $p : \mathbb{R}^n \to \mathbb{R}^m$ is an affine function of x, $s : \mathbb{R}^n \to \mathbb{R}$, and $P(x) : \mathbb{R}^n \to \mathbb{R}^{m \times m}$, and finally where

$$\mathcal{Z} := \{z \in \mathbb{R}^m \,|\, z^T Q z \leq r\}\ ,$$

with $Q \in \mathbb{R}^{m \times m}$, $q \in \mathbb{R}^m$, and $r \in \mathbb{R}$.

- S-lemma can be used to demonstrate that this RC is equivalent to

$$\exists\, \lambda \geq 0 \qquad \begin{bmatrix} P(x) + \lambda Q & -p(x)/2 \\ -p(x)^T/2 & -s(x) - r\lambda \end{bmatrix} \succeq 0$$

even when P(x) is not PSD as long as $\bar{z}^T Q \bar{z} < r$ for some $\bar{z}$.

Some useful theorems

Theorem 6.5. : *If* $\mathcal{Z} := \{z \in \mathbb{R}^m \,|\, 0 \leq z \leq 1,\ \sum_i z_i \leq \rho\}$*, then*

$$\begin{aligned} \delta^*(v|\mathcal{Z}) := \quad & \inf_{\omega \in \mathbb{R}^m, \lambda \in \mathbb{R}} && \sum_i \omega_i + \rho\lambda \\ & \text{subject to} && \lambda \geq v_i - \omega_i \,,\ \forall\, i \\ & && \lambda \geq 0,\ \omega \geq 0\ . \end{aligned}$$

Theorem 6.7. : *If* $\mathcal{Z} \subset \mathbb{R}^m$ *is an affine projection of* $\mathcal{Z}_1 \subset \mathbb{R}^{m_1}$*, namely that* $\mathcal{Z} := \{z \in \mathbb{R}^m \,|\, \exists z' \in \mathcal{Z}_1,\ z = Az' + a_0\}$ *for some* $A \in \mathbb{R}^{m \times m_1}$ *and* $a_0 \in \mathbb{R}^m$*, then* $\delta^*(v|\mathcal{Z}) = a_0^T v + \delta^*(A^T v|\mathcal{Z}_1)$*.*

Theorem 6.9. : *If* $g(x, z)$ *is a positive affine mapping of* $g'(x, z)$*, namely that* $g(x, z) := \alpha g'(x, z) + \beta$ *for some* $\alpha > 0$*, then* $g_*(x, v) = \alpha g'_*(x, v/\alpha) - \beta$*.*

Other useful theorems

Corollary 6.8. : *Consider using the budgeted uncertainty set* $\mathcal{Z} := \{z \in \mathbb{R}^m \mid -1 \leq z \leq 1,\ \sum_i |z_i| \leq \Gamma\}$, *then*

$$\begin{aligned}
\delta^*(v|\mathcal{Z}) := \quad & \inf_{\omega^+ \in \mathbb{R}^m, \omega^- \in \mathbb{R}^m, \lambda \in \mathbb{R}} && \sum_i \omega_i^+ + \sum_i \omega_i^- + \rho\lambda \\
& \text{subject to} && \lambda \geq v_i - \omega_i^+ \,,\ \forall\, i \\
& && \lambda \geq -v_i - \omega_i^- \,,\ \forall\, i \\
& && \omega \geq 0,\ \lambda \geq 0\ .
\end{aligned}$$

Hence, the robust counterpart takes the form:

$$\exists \omega^+ \in \mathbb{R}^m, \omega^- \in \mathbb{R}^m, \lambda \in \mathbb{R}, v \in \mathbb{R}^m, \quad \begin{cases} \sum_i \omega_i^+ + \sum_i \omega_i^- + \rho\lambda - g_*(x, v) \leq 0 \\ \lambda \geq v_i - \omega_i^+ \,,\ \forall\, i \\ \lambda \geq -v_i - \omega_i^- \,,\ \forall\, i \\ \omega^+ \geq 0,\ \omega^- \geq 0,\ \lambda \geq 0 \end{cases} .$$

Table 6.1: Table of reformulations for uncertainty sets (Table 1 in [8])

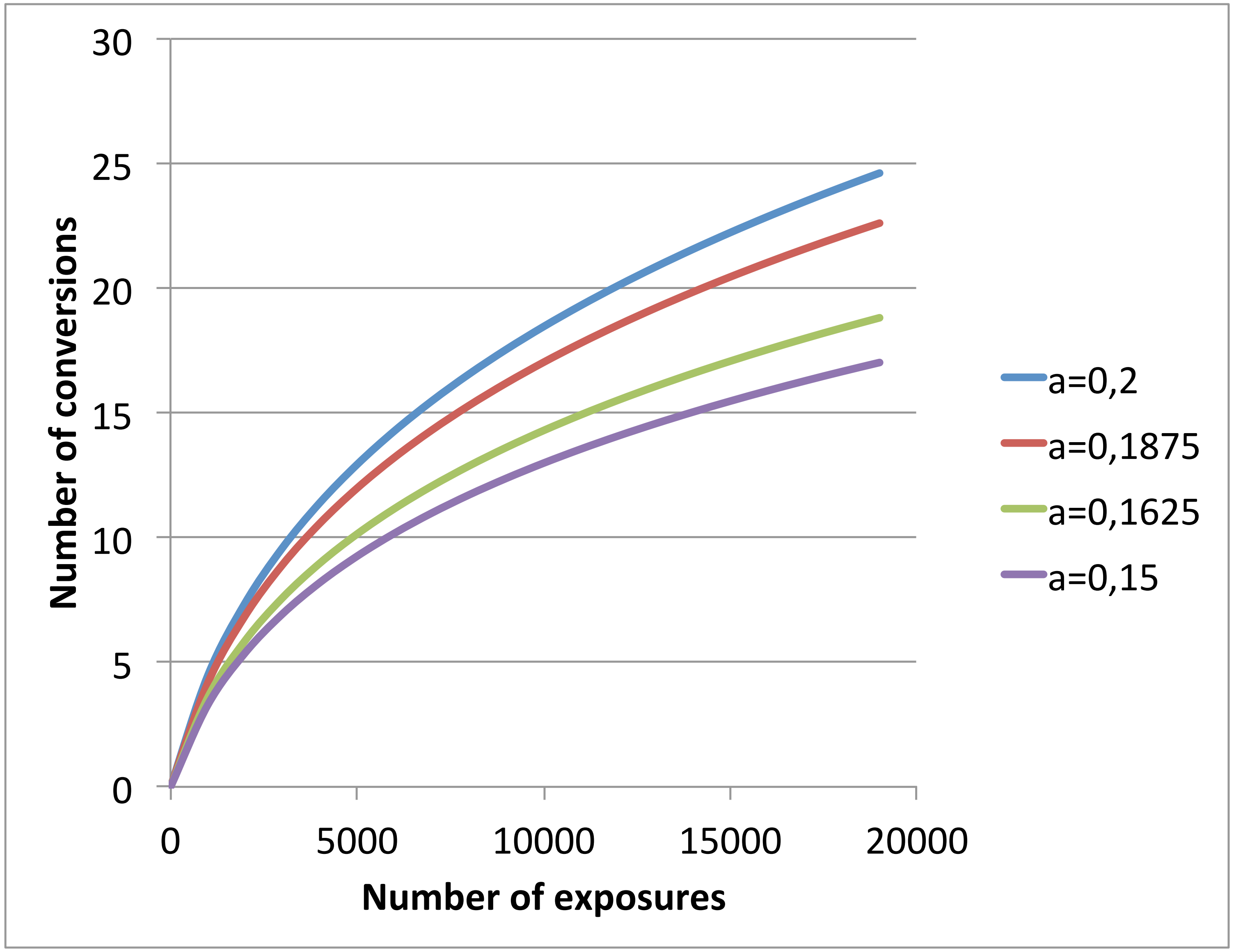
Uncertainty region	$\mathcal{Z}$	Support function $\delta^*(v\|\mathcal{Z})$
Box	$\\|z\\|_\infty \leq \rho$	$\rho\\|v\\|_1$
Ball	$\\|z\\|_2 \leq \rho$	$\rho\\|v\\|_2$
Polyhedral	$b - Bz \geq 0$	$\inf_{w\geq 0: B^T w = v} b^T w$
Cone	$b - Bz \in C$	$\inf_{w \in C^*: B^T w = v} b^T w$
KL-Divergence	$\sum_l z_l \ln\left(\frac{z_l}{z_l^0}\right) \leq \rho$	$\inf_{u\geq 0} \sum_l z_l^0 u e^{(v_l/u)-1} + \rho u$
Geometric prog.	$\sum_i \alpha_i e^{(d_i)^T z} \leq \rho$	$\inf_{u\geq 0,\, w\geq 0: \sum_i d_i w_i = v} \sum_i \{w_i \ln\left(\frac{w_i}{\alpha_i u}\right) - w_i\} + \rho u$
Intersection	$\mathcal{Z} = \cap_i \mathcal{Z}_i$	$\inf_{\{w_i\}: \sum_i w_i = v} \sum_i \delta^*(w^i\|\mathcal{Z}_i)$
Example	$\mathcal{Z}_k = \{z\|\\|z\\|_k \leq \rho_k\}$ $k = 1, 2$	$\inf_{(w^1,w^2): w^1+w^2=v} \rho_1\\|w^1\\|_\infty + \rho_2\\|w^2\\|_2$
Minkowski sum	$\mathcal{Z} = \mathcal{Z}_1 + \cdots + \mathcal{Z}_K$	$\sum_i \delta^*(v\|\mathcal{Z}_i)$
Example	$\mathcal{Z}_1 = \{z \mid \\|z\\|_\infty \leq \rho_\infty\}$ $\mathcal{Z}_2 = \{z \mid \\|z\\|_2 \leq \rho_2\}$	$\rho_\infty\\|v\\|_1 + \rho_2\\|v\\|_2$
Convex hull	$\mathcal{Z} = \mathrm{conv}(\mathcal{Z}_1, \ldots, \mathcal{Z}_K)$	$\max_i \delta^*(v\|\mathcal{Z}_i)$
Example	$\mathcal{Z}_1 = \{z \mid \\|z\\|_\infty \leq \rho_\infty\}$ $\mathcal{Z}_2 = \{z \mid \|z - z^0\\|_2 \leq \rho_2\}$	$\max\{\rho_\infty\\|v\\|_1, (z^0)^T v + \rho_2\\|v\|\|_2\}$

Table 6.2: Table of reformulations for constraint functions (Table 2 in [8])

Constraint function	$g(x,z)$	Partial concave conjugate $g_*(x,v)$
Linear in z	$z^T g(x)$	$\begin{cases} 0 & \text{if } v = g(x) \\ -\infty & \text{otherwise} \end{cases}$
Concave in z, separable in z and x	$g(z)^T x$	$\sup_{\{s^i\}_{i=1}^n : \sum_{i=1}^n s^i = v} \sum_i x_i (g_i)_*(s^i / x_i)$
Example	$-\sum_i \frac{1}{2}(z^T Q_i z) x_i$	$\sup_{\{s^i\}_{i=1}^n : \sum_{i=1}^n s^i = v} -\frac{1}{2} \sum_{i=1}^n \frac{(s^i)^T Q_i^{-1} s^i}{x_i}$
Sum of functions	$\sum_i g_i(x,z)$	$\sup_{\{s^i\}_{i=1}^n : \sum_i s^i = v} \sum_i (g_i)_*(x, s^i)$
Sum of separable functions	$\sum_i g_i(x, z_i)$	$\sum_{i=1}^n (g_i)_*(x, v_i)$
Example	$-\sum_{i=1}^m x_i^{z_i}$, $x_i > 1, 0 \leq z \leq 1$	$\begin{cases} \sum_{i=1}^m \left(\frac{v_i}{\ln x_i} \ln \frac{-v_i}{\ln x_i} - \frac{v_i}{\ln x_i} \right) & \text{if } v \leq 0 \\ -\infty & \text{otherwise} \end{cases}$
Exponential in z	$-g(x)e^z$ with $g(x) > 0, \forall x$	$\begin{cases} v \ln(-v/g(x)) - v & \text{if } v \leq 0 \\ -\infty & \text{otherwise} \end{cases}$

Planning an ad campaign with exposure rate uncertainty

Figure of the converted number of customers per ad displayed on a website according to $h_i(x_i) := 30(1 + x_i/1000)^{a_i} - 30$



Planning an ad campaign with exposure rate uncertainty

- Derive a tractable reformulation for the robust counterpart of this problem:

$$\begin{aligned} \underset{x}{\text{maximize}} \quad & \sum_i h_i(x) \\ \text{subject to} \quad & \sum_i p_i x_i \leq B \\ & x \geq 0 \ , \end{aligned}$$

with

$$h_i(x_i) := c_i(1+x_i/d_i)^{a_i} - c_i$$

$$\mathcal{U}_1 := \{a \in \mathbb{R}^n \,|\, \exists z \in \mathbb{R}^n, 0 \leq z \leq 1, \sum_i z_i \leq \Gamma,\, a_i = \bar{a}_i(1 - 0.25 z_i)\,,\, \forall i\}$$

where $\bar{a} \in \ [0,\ 1]^n$

Exercise

Exercise 6.3. ***(More robust non-linear reformulations)***

Consider the robust optimization problem:

$$\begin{aligned} \underset{x}{\text{maximize}} \quad & \min_{z \in \mathcal{Z}} \sum_i x_i \exp(z_i) \\ \text{subject to} \quad & \sum_i x_i \leq 1 \\ & x \geq 0 \; , \end{aligned}$$

where

$$\mathcal{Z} := \{ z \in \mathbb{R}^n \,|\, \exists v \in [-1,1]^n, w \in [-1,1], z = \mu + Q(v + w), \|v\|_1 \leq \Gamma \} \; .$$

Question: Derive a tractable reformulation of this problem as a convex optimization problem of finite dimension ?