Erick's notes

I wish to warmly thank Daniel Kuhn for sharing this keynote presentation

Data-Driven Distributionally Robust Optimization using the Wasserstein Metric

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Stochastic Programming

$$\text{SP}: \quad \min_{x \in \mathcal{X}} \mathbb{E}^{\mathbb{P}}\left[\ell(x, \xi)\right]$$

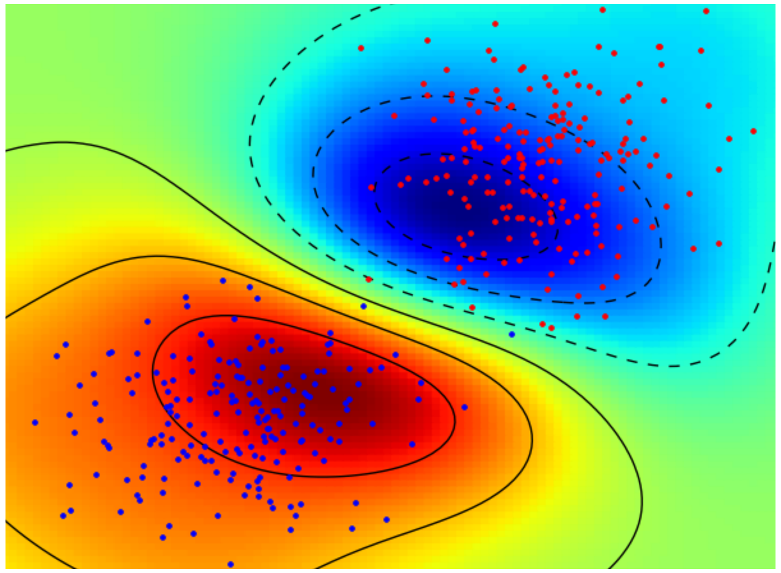
$$J^* = \min \text{SP}$$

$$x^* = \operatorname{argmin} \text{SP}$$

Applications:

Supply Chain Mgmt.

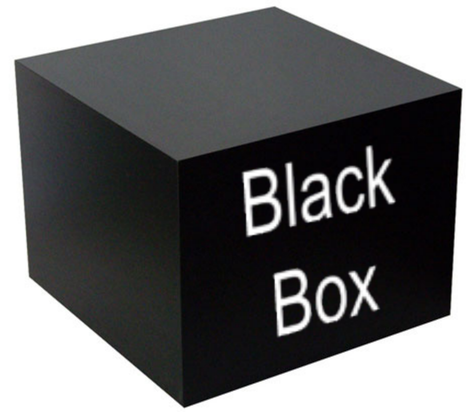
Portfolio Mgmt.



Machine Learning

Data-Driven Stochastic Programming

$$\underbrace{\widehat{\xi}_1, \widehat{\xi}_2, \ldots, \widehat{\xi}_N}_{\text{training data} \sim \mathbb{P}^N} \longrightarrow$$



$$\longrightarrow \begin{cases} \widehat{x} & \text{decision} \\ \widehat{J} & \text{certificate} \end{cases}$$

$$\underbrace{\widehat{\xi}_1, \widehat{\xi}_2, \ldots, \widehat{\xi}_M}_{\text{test data} \sim \mathbb{P}^M} \longrightarrow \frac{1}{M}\sum_{i=1}^{M} \ell(\widehat{x}, \widehat{\xi}_i) \to \underbrace{\mathbb{E}^{\mathbb{P}}\left[\ell(\widehat{x}, \xi)\right]}_{\text{out-of-sample performance}} \geq J^*$$

Aim: establish a finite sample guarantee

$$\mathbb{P}^N \left\{ \mathbb{E}^{\mathbb{P}}\left[\ell(\widehat{x}, \xi)\right] \leq \widehat{J} \right\} \geq 1 - \beta$$

Sample Average Approximation (SAA)

$$\text{SAA}: \quad \min_{x \in \mathcal{X}} \frac{1}{N} \sum_{i=1}^{N} \ell(x, \widehat{\xi}_i)$$

$$\underbrace{\widehat{\xi}_1, \widehat{\xi}_2, \ldots, \widehat{\xi}_N}_{\text{training data} \sim \mathbb{P}^N} \longrightarrow$$

$$\longrightarrow \begin{cases} \widehat{x} = \text{argmin}\,(\text{SAA}) \\ \widehat{J} = J^* + \varepsilon \end{cases}$$

Finite sample guarantee:[2)]

$$\mathbb{P}^N \left\{ \mathbb{E}^{\mathbb{P}} \left[\ell(\widehat{x}, \xi) \right] \leq \widehat{J} \right\} \geq 1 - \beta \quad \text{if}$$

$$N \geq C \left(\frac{\text{diam}(\mathcal{X})\, \text{lip}(\ell)}{\varepsilon} \right)^2 \left[\text{dim}(x) \log \left(\frac{\text{diam}(\mathcal{X})\, \text{lip}(\ell)}{\varepsilon} \right) + \log \left(\frac{c}{\beta} \right) \right]$$

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2) Shapiro & Nemirovski, *Springer*, 2005.

SAA with Scarce Data

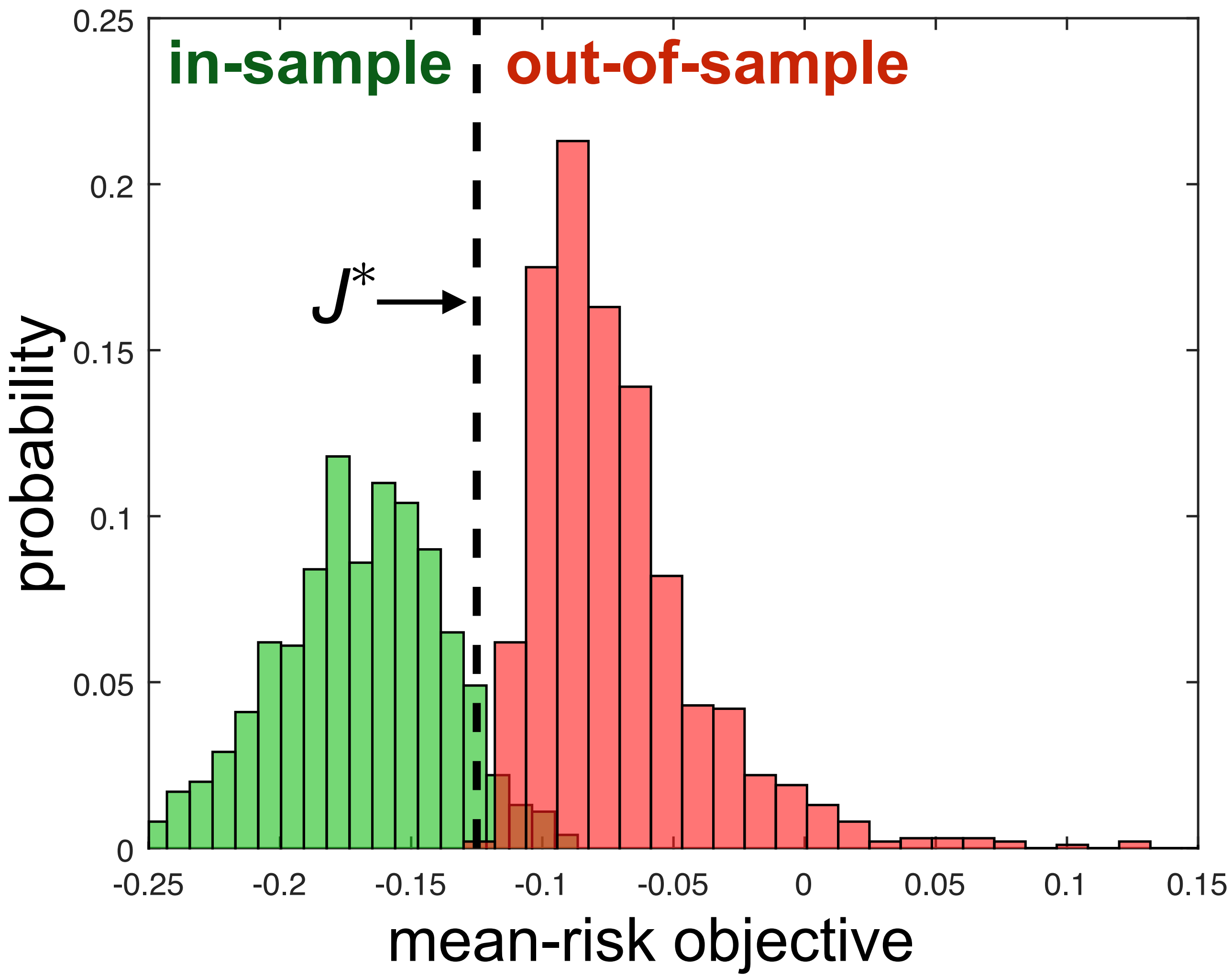
Mean-risk portfolio problem

$$\min_{x \in \mathcal{X}} \left\{ \mathbb{E}^{\mathbb{P}}\left[-x^\top \xi \right] + \rho\, \mathbb{P}\text{-CVaR}_\alpha(-x^\top \xi) \right\}$$

- 10 assets
- $\rho = 10$
- $\alpha = 20\%$
- $\xi_i = \psi + \zeta_i$ where $\psi \sim \mathcal{N}(0, 2\%)$ and $\zeta_i \sim \mathcal{N}(i \times 3\%, i \times 2.5\%)$

Erick's notes
In our notation
1-epsilon=80%

Performance of SAA solution



- 30 training samples
- in-sample: optimistic bias
- out-of-sample: pessimistic bias

Distributionally Robust Optimization (DRO)

$$\text{DRO}: \quad \min_{x \in \mathcal{X}} \ \sup_{\mathbb{Q} \in \widehat{\mathcal{P}}_N} \ \mathbb{E}^{\mathbb{Q}}\left[\ell(x, \xi)\right]$$

$$\underbrace{\widehat{\xi}_1, \widehat{\xi}_2, \ldots, \widehat{\xi}_N}_{\text{training data} \sim \mathbb{P}^N} \longrightarrow$$

$$\longrightarrow \begin{cases} \widehat{x} = \text{argmin (DRO)} \\ \widehat{J} = \min \text{(DRO)} \end{cases}$$

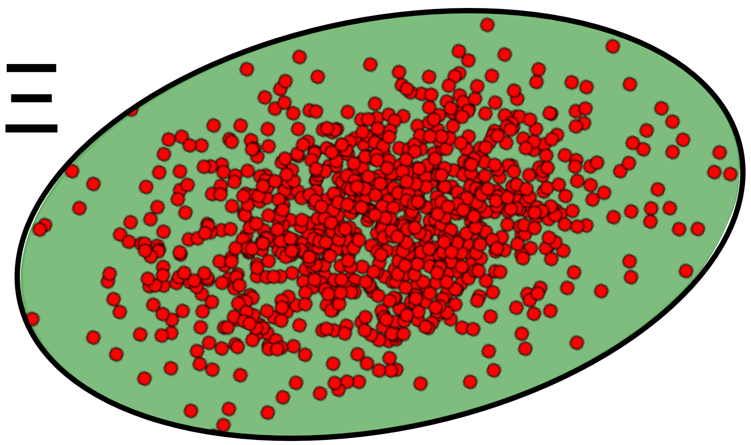
Desired properties:[3]

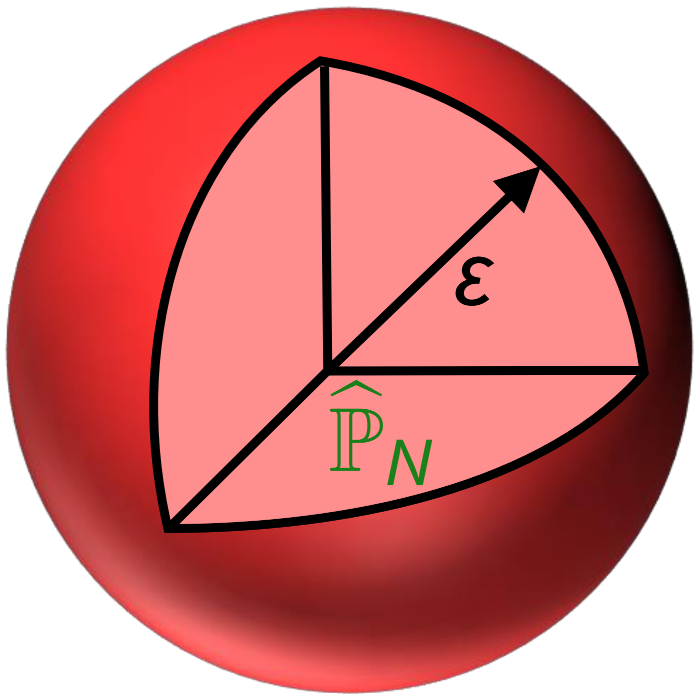
- Finite sample guarantee: $\mathbb{P}^N \left\{ \mathbb{E}^{\mathbb{P}} \left[\ell(\widehat{x}, \xi)\right] \leq \widehat{J} \right\} \geq 1 - \beta$
- Asymptotic guarantee: $\mathbb{P}^\infty \left\{ \lim_{N \to \infty} \widehat{x} = x^* \right\} = 1$
- Tractability: DRO is in the same complexity class as SAA

3) Bertsimas, Gupta & Kallus, *http://arxiv.org*, 2014.

Wasserstein Ambiguity Set[4)]

$$\mathbb{B}_\varepsilon(\widehat{\mathbb{P}}_N) = \left\{\mathbb{Q} : \mathbb{Q}(\xi \in \Xi) = 1\right\} \cap \left\{\mathbb{Q} : d_{\mathrm{W}}(\widehat{\mathbb{P}}_N, \mathbb{Q}) \leq \varepsilon\right\}$$





Empirical distribution: $\widehat{\mathbb{P}}_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\widehat{\xi}_i}$

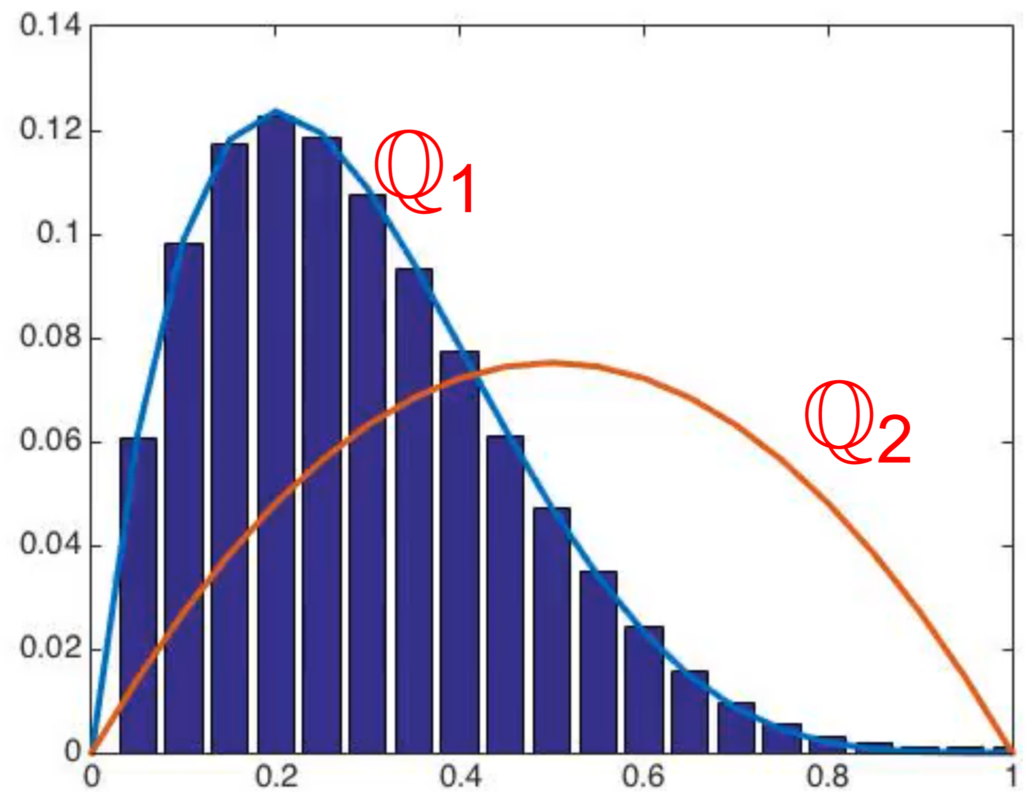
Wasserstein metric: $d_{\mathrm{W}}(\mathbb{Q}_1, \mathbb{Q}_2) = \sup_{\mathrm{lip}(f)\leq 1} \mathbb{E}^{\mathbb{Q}_1}[f(\xi)] - \mathbb{E}^{\mathbb{Q}_2}[f(\xi)]$

4) Pflug & Wozabal, *Quant. Finance*, 2007.

Kantorovich-Rubinstein Theorem[6)]

LP duality implies:

$$d_{\mathrm{W}}(\mathbb{Q}_1, \mathbb{Q}_2) = \text{inf} \left\{ \mathbb{E}^{\mathbb{Q}} \left[\|\xi_1 - \xi_2\| \right] : \begin{array}{l} \mathbb{Q} \text{ distribution of } \xi_1 \text{ and } \xi_2 \\ \text{with marginals } \mathbb{Q}_1 \text{ and } \mathbb{Q}_2 \end{array} \right\}$$



$d_{\mathrm{W}}(\mathbb{Q}_1, \mathbb{Q}_2) =$ minimum cost of moving $\mathbb{Q}_1$ to $\mathbb{Q}_2$

6) Kantorovich & Rubinstein, *Vestn. Lenin. U.*, 1958.

Finite-Sample Guarantee

Measure concentration theorem:[5)] If the tail of $\mathbb{P}$ decays exponentially at rate a, then

$$\mathbb{P}^N\left\{d_{\mathrm{W}}(\mathbb{P},\widehat{\mathbb{P}}_N) > \varepsilon\right\} \leq \begin{cases} Ce^{-cN\varepsilon^{\dim(\xi)}} & \text{if } \varepsilon \leq 1, \\ Ce^{-cN\varepsilon^{a}} & \text{if } \varepsilon > 1. \end{cases}$$

$$\varepsilon_N(\beta) = \begin{cases} (\log(C\beta^{-1})/(c\,N))^{\frac{1}{a}} & \text{if } N < \log(C\beta^{-1})/c \\ (\log(C\beta^{-1})/(c\,N))^{\frac{1}{\dim(\xi)}} & \text{if } N \geq \log(C\beta^{-1})/c \end{cases}$$

$$\Longrightarrow \quad \mathbb{P}^N\left\{\mathbb{P} \in \mathbb{B}_{\varepsilon_N(\beta)}(\widehat{\mathbb{P}}_N)\right\} \geq 1-\beta$$

$$\Longrightarrow \quad \mathbb{P}^N\left\{\mathbb{E}^{\mathbb{P}}\left[\ell(\widehat{x},\xi)\right] \leq \widehat{J}\right\} \geq 1-\beta$$

[5)] Fournier & Guillin, *Probab. Theory Rel.*, 2014.

Asymptotic Guarantee

Convergence theorem: If the tail of $\mathbb{P}$ decays exponentially and the Wasserstein radius is $\varepsilon_N(\beta_N)$ with $\beta_N \propto \exp(-N^\delta)$ for some $\delta > 0$, then we have:

- Convergence of optimal values:

$$\mathbb{P}^\infty \left\{ \lim_{N\to\infty} \widehat{J}_N = J^* \right\} = 1$$

- Convergence of optimal solutions:

$$\mathbb{P}^\infty \left\{ \limsup_{N\to\infty} \{\widehat{x}_N\} \subseteq \arg\min(\mathrm{SP}) \right\} = 1$$

Tractability

Convexity assumption:

- $-\ell$ is proper, convex and lsc;
- Ξ is convex and closed.

Tractability

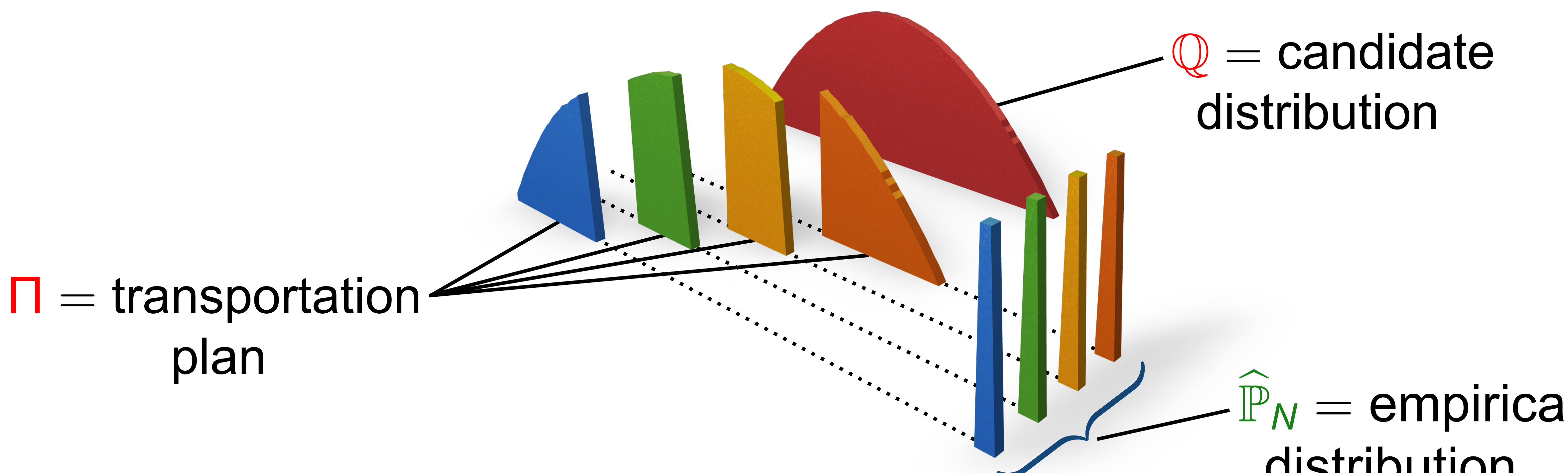
Worst-case expectation problem:

$$\sup_{\mathbb{Q} \in \widehat{\mathcal{P}}_N} \mathbb{E}^{\mathbb{Q}} \left[\ell(\boldsymbol{\xi}) \right]$$

Tractability

Use the Kantorovich-Rubinstein theorem:

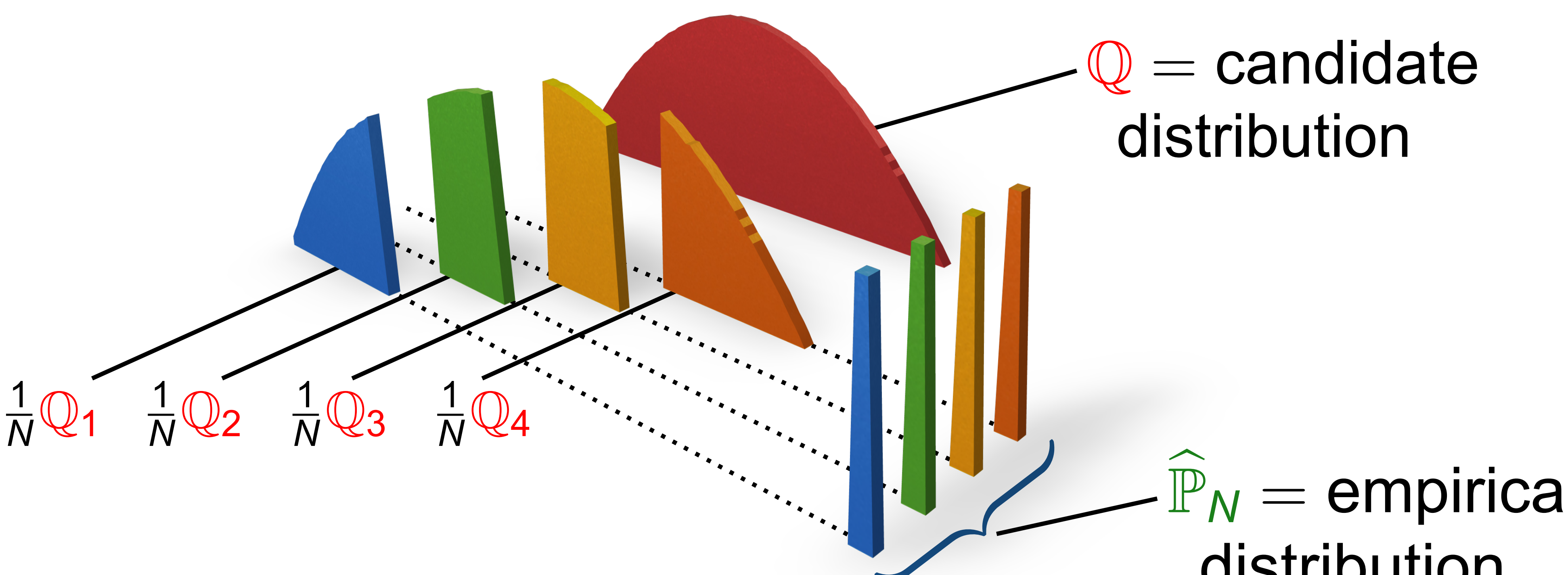
$$\begin{aligned} \sup_{\Pi, \mathbb{Q}} \quad & \int_{\Xi} \ell(\xi)\, \mathbb{Q}(\mathrm{d}\xi) \\ \text{s.t.} \quad & \int_{\Xi^2} \|\xi - \xi'\|\, \Pi(\mathrm{d}\xi, \mathrm{d}\xi') \le \varepsilon \\ & \left\{ \begin{array}{l} \Pi \text{ is a joint distribution of } \xi \text{ and } \xi' \\ \text{with marginals } \mathbb{Q} \text{ and } \widehat{\mathbb{P}}_N \text{, respectively} \end{array} \right. \end{aligned}$$



Tractability

Decompose Π into $\mathbb{Q}_1, \ldots, \mathbb{Q}_N$:

$$\begin{aligned} \sup_{\mathbb{Q}_i} \quad & \frac{1}{N} \sum_{i=1}^{N} \int_{\Xi} \ell(\xi)\, \mathbb{Q}_i(\mathrm{d}\xi) \\ \text{s.t.} \quad & \frac{1}{N} \sum_{i=1}^{N} \int_{\Xi} \|\xi - \widehat{\xi}_i\|\, \mathbb{Q}_i(\mathrm{d}\xi) \leq \varepsilon \end{aligned}$$



Dual of the moment problem is a robust program:

$$
\begin{aligned}
\inf_{\lambda, s_i} \quad & \lambda \varepsilon + \frac{1}{N} \sum_{i=1}^{N} s_i \\
\text{s.t.} \quad & \ell(\xi) - \lambda \|\xi - \widehat{\xi}_i\| \leq s_i \quad \forall \xi \in \Xi, \quad \forall i \leq N \\
& \lambda \geq 0
\end{aligned}
$$

Introduce the indicator function of Ξ:

$$\begin{aligned} \inf_{\lambda, s_i} \quad & \lambda \varepsilon + \frac{1}{N} \sum_{i=1}^{N} s_i \\ \text{s.t.} \quad & \ell(\xi) - \lambda \|\xi - \widehat{\xi}_i\| - \delta_{\Xi}(\xi) \le s_i \quad \forall \xi \in \mathbb{R}^m, \quad \forall i \le N \\ & \lambda \ge 0 \end{aligned}$$

Reformulate robust program as bilevel program:

$$\begin{aligned}
\inf_{\lambda, s_i} \quad & \lambda \varepsilon + \frac{1}{N} \sum_{i=1}^{N} s_i \\
\text{s.t.} \quad & \sup_{\xi \in \mathbb{R}^m} \left(\ell(\xi) - \lambda \|\xi - \widehat{\xi}_i\| - \delta_{\Xi}(\xi) \right) \le s_i \quad \forall i \le N \\
& \lambda \ge 0
\end{aligned}$$

Tractability

Take the Fenchel dual of the lower-level problem:[7)]

$$
\begin{aligned}
\inf_{\lambda, s_i, v_i, z_i} \quad & \lambda \varepsilon + \frac{1}{N} \sum_{i=1}^{N} s_i \\
\text{s.t.} \quad & \boxed{[-\ell]^*}(z_i - v_i) + \boxed{\sigma_\Xi}(v_i) - z_i{}^\top \widehat{\xi}_i \le s_i \quad \forall i \le N \\
& \boxed{\|z_i\|_*} \le \lambda
\end{aligned}
$$

dual norm of z_i

convex conjugate of $-\ell$

support function of Ξ

7) Ben-Tal, den Hertog & Vial, *Math. Program.*, 2015.

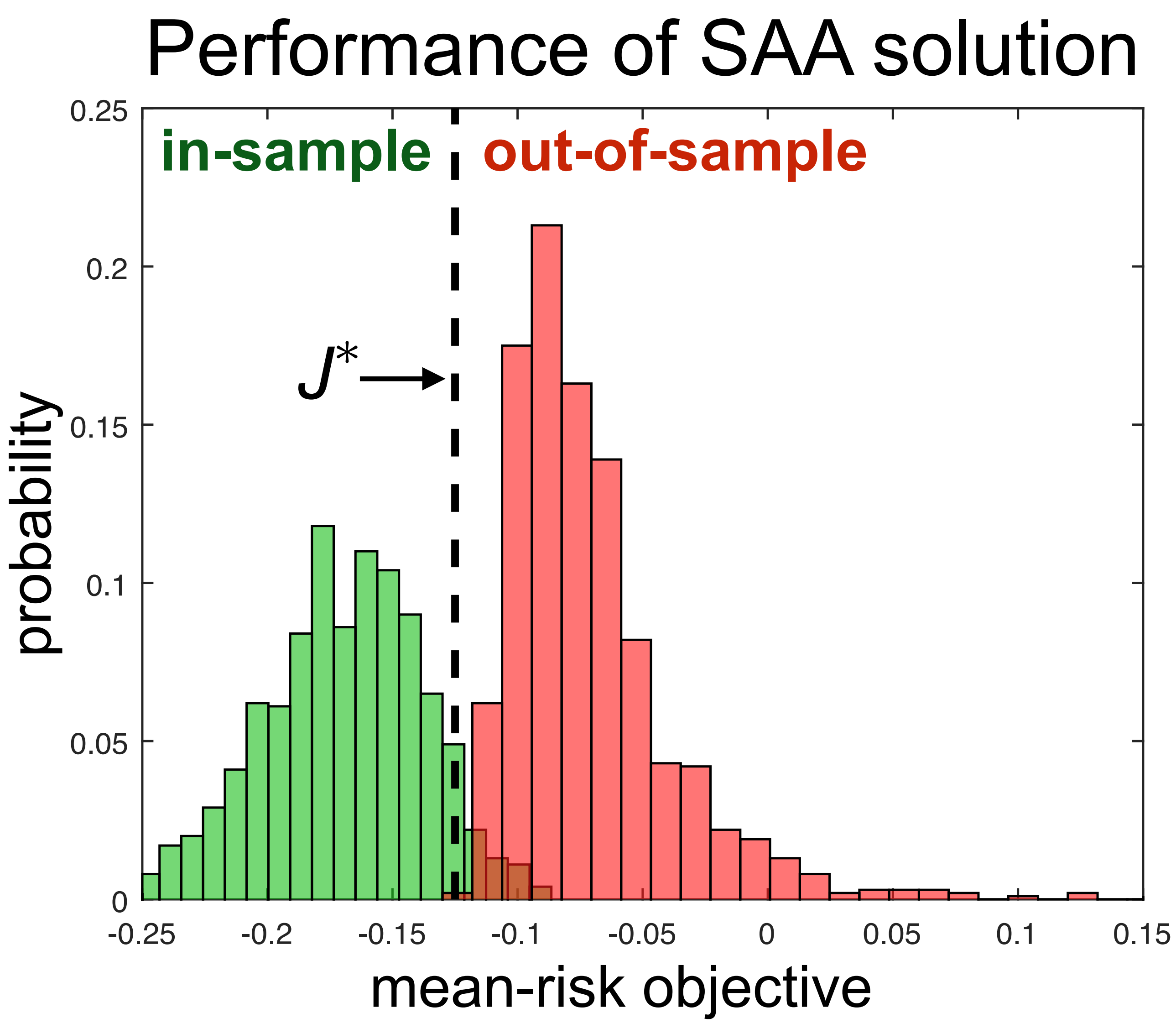
Tractability

The worst-case expectation equals:

$$\begin{aligned} \inf_{\lambda, s_i, v_i, z_i} \quad & \lambda \varepsilon + \frac{1}{N} \sum_{i=1}^{N} s_i \\ \text{s.t.} \quad & [-\ell]^*(z_i - v_i) + \sigma_{\Xi}(v_i) - z_i{}^\top \widehat{\xi}_i \leq s_i \quad \forall i \leq N \\ & \|z_i\|_* \leq \lambda \end{aligned}$$

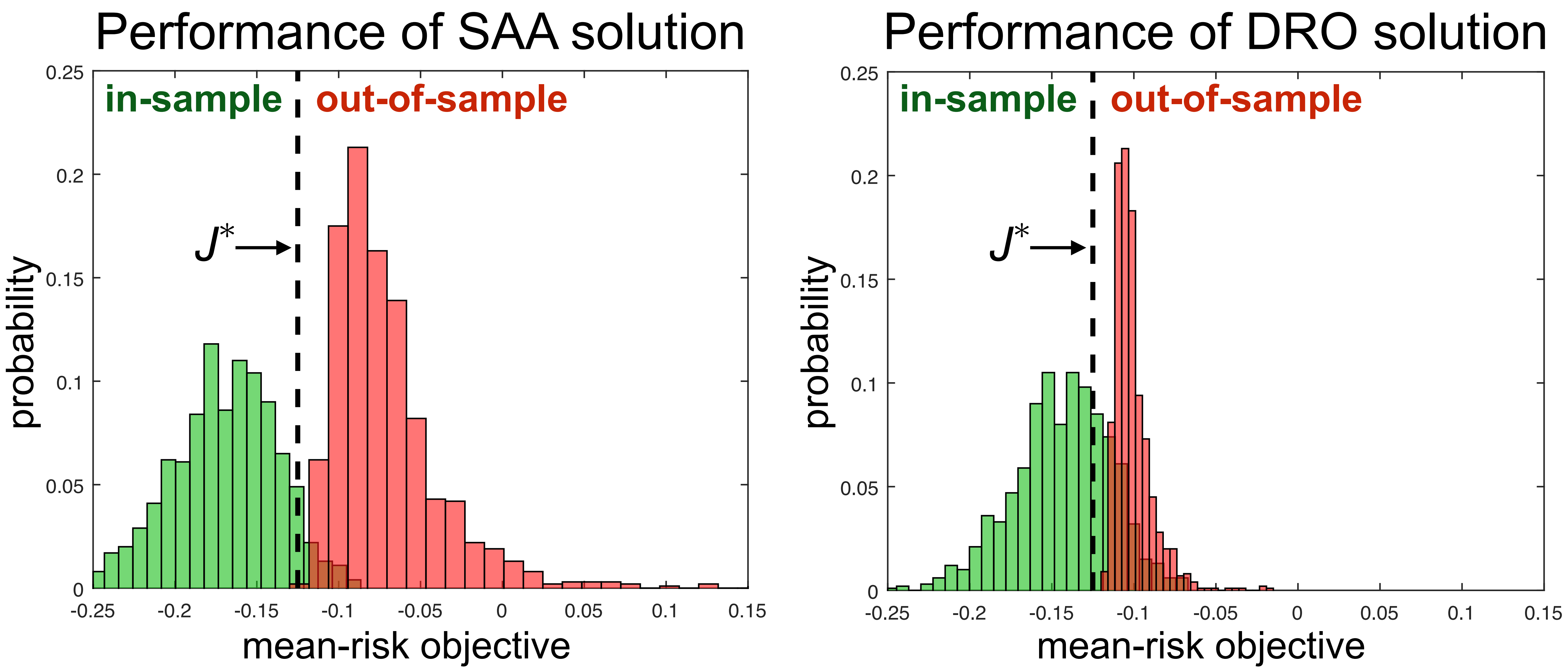
- **Finite convex program**
- Problem size grows **polynomially** in input data
- Can be combined with minimization over $x \in \mathcal{X}$: the resulting problem is in the **same complexity class as SAA**

DRO with Scarce Data



- in-sample: optimistic bias
- out-of-sample: pessimistic bias

DRO with Scarce Data



- in-sample: optimistic bias
- out-of-sample: pessimistic bias
- **DRO reduces bias & post-decision disappointment**

Worst-Case Distributions

Dualize finite convex program:

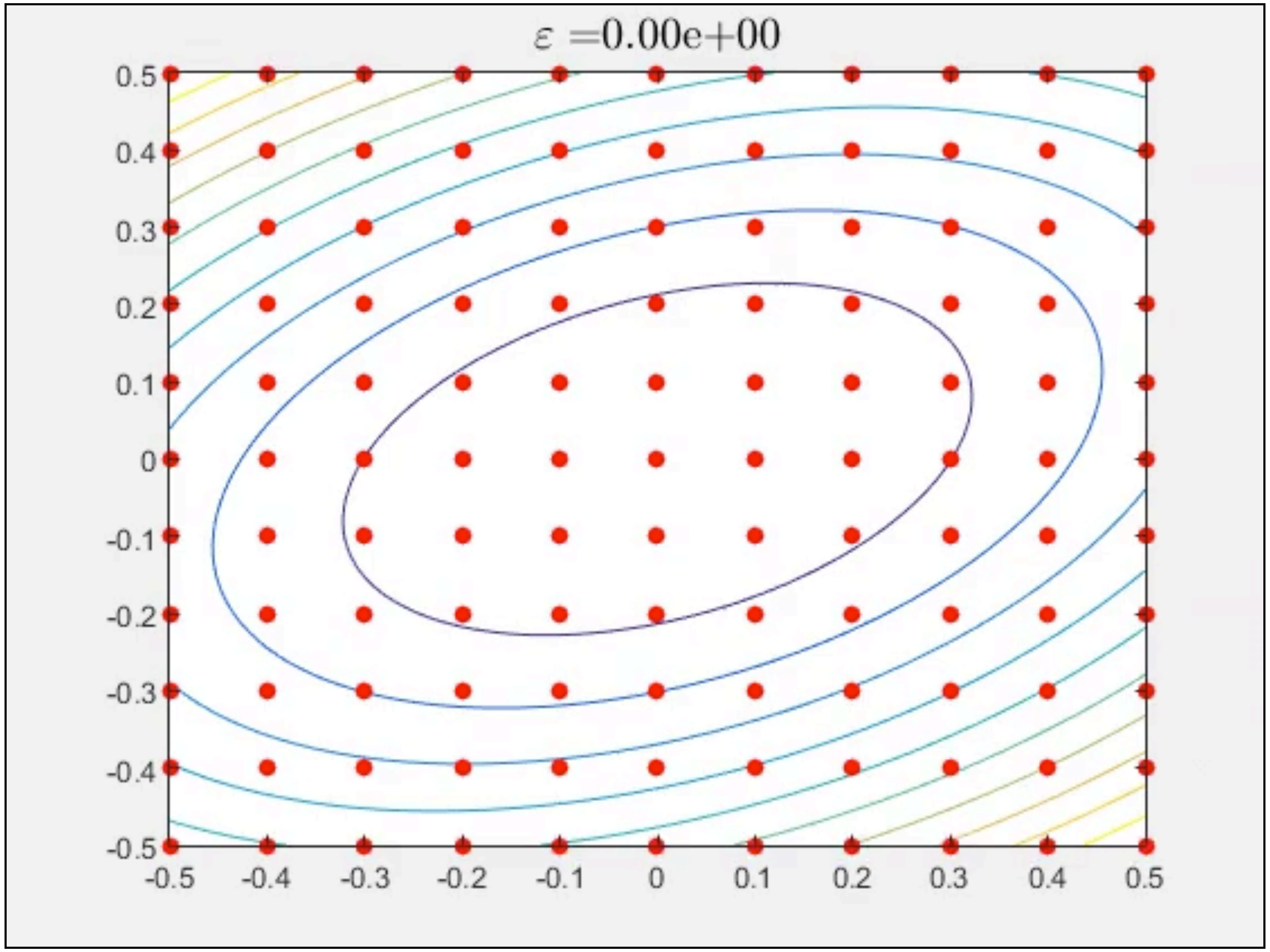
$$\sup_{\mathbb{Q}\in\widehat{\mathcal{P}}_N} \mathbb{E}^{\mathbb{Q}}\left[\ell(\xi)\right] \quad = \quad \begin{array}{cl} \sup\limits_{q_i} & \dfrac{1}{N}\displaystyle\sum_{i=1}^{N} \ell(\widehat{\xi}_i - q_i) \\ \text{s.t.} & \dfrac{1}{N}\displaystyle\sum_{i=1}^{N} \|q_i\| \leq \varepsilon \\ & \widehat{\xi}_i - q_i \in \Xi \quad \forall i \leq N \end{array}$$

If $q_1^\star, \ldots, q_N^\star$ solves the dual problem, then

$$\mathbb{Q}^\star = \frac{1}{N}\sum_{i=1}^{N} \delta_{\widehat{\xi}_i - q_i^\star}$$

is a worst-case distribution.

Concave Quadratic Loss Function



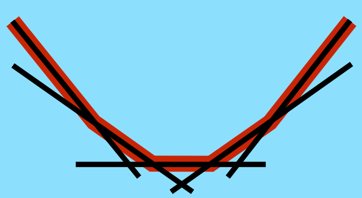
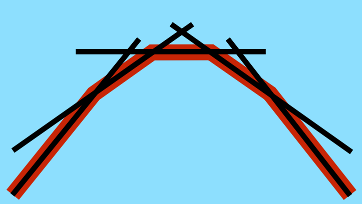
Generalized Loss Functions

Generalized convexity assumption:

- $\ell = \max\{\ell_1, \ldots, \ell_J\}$ where each $-\ell_j$ is proper, convex and lsc.

Maxima & Minima of Affine Functions

Set $\Xi = \{\xi : C\xi \leq d\}$ and $a_j(\xi) = a_j^\top \xi - b_j$ for all $j \leq J$.

$\ell(\xi)$	$\sup_{\mathbb{Q} \in \widehat{\mathcal{P}}_N} \mathbb{E}^{\mathbb{Q}}[\ell(\xi)]$
$\max_{j \leq J} a_j(\xi)$ 	$\begin{array}{lll} \inf_{\lambda, s_i, \gamma_{ij}} & \lambda\varepsilon + \frac{1}{N}\sum_{i=1}^{N} s_i & \\ \text{s.t.} & b_j + a_j^\top \widehat{\xi}_i + \gamma_{ij}^\top (d - C\widehat{\xi}_i) \leq s_i & \forall i \leq N,\ \forall j \leq J \\ & \lVert C^\top \gamma_{ij} - a_j \rVert_* \leq \lambda & \forall i \leq N,\ \forall j \leq J \\ & \gamma_{ij} \geq 0 & \forall i \leq N,\ \forall j \leq J \end{array}$
$\min_{j \leq J} a_j(\xi)$ 	$\begin{array}{lll} \inf_{\lambda, s_i, \gamma_i, \theta_i} & \lambda\varepsilon + \frac{1}{N}\sum_{i=1}^{N} s_i & \\ \text{s.t.} & \theta_i^\top (b - A\widehat{\xi}_i) + \gamma_i^\top (d - C\widehat{\xi}_i) \leq s_i & \forall i \leq N \\ & \lVert C^\top \gamma_i - A^\top \theta_i \rVert_* \leq \lambda & \forall i \leq N \\ & \mathrm{e}^\top \theta_i = 1, \quad \gamma_i \geq 0, \quad \theta_i \geq 0 & \forall i \leq N \end{array}$

Two-Stage Stochastic Programming

Objective uncertainty:

$$\ell(\xi) = \underbrace{\inf_{y} \left\{ y^\top Q \xi \ : \ W y \geq h \right\}}_{\text{concave piecewise affine}}$$

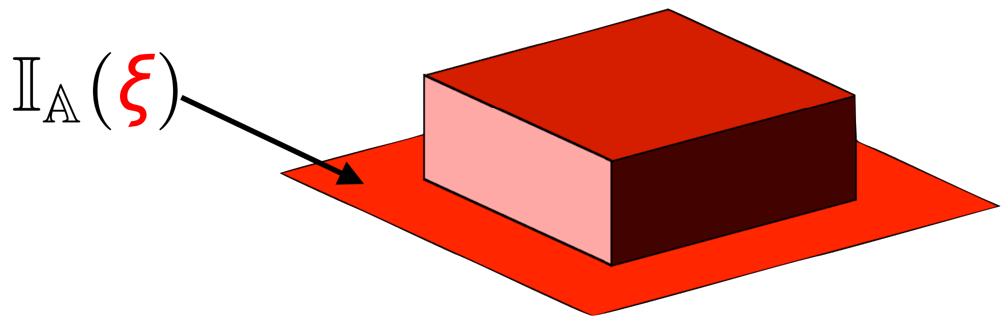
Constraint uncertainty:

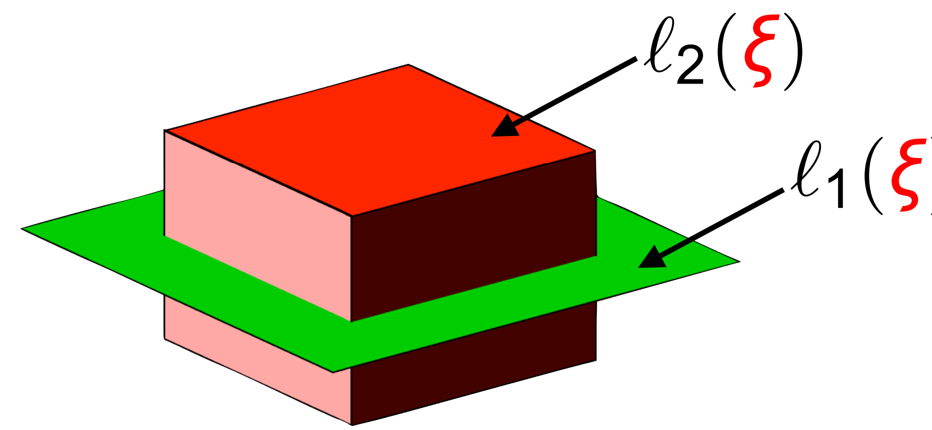
$$\ell(\xi) = \underbrace{\inf_{y} \left\{ q^\top y \ : \ W y \geq H \xi + h \right\}}_{\text{convex piecewise affine}}$$

Uncertainty Quantification

Probability of being inside a polytope:

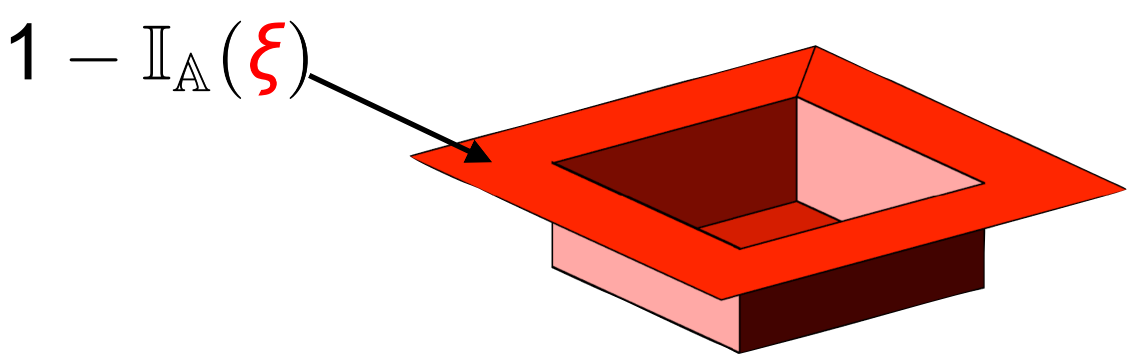
$$\sup_{\mathbb{Q}\in\widehat{\mathcal{P}}_N} \mathbb{Q}\left[\xi \in \mathbb{A}\right] = \sup_{\mathbb{Q}\in\widehat{\mathcal{P}}_N} \mathbb{E}^{\mathbb{Q}}\left[\mathbb{I}_{\mathbb{A}}(\xi)\right] = \sup_{\mathbb{Q}\in\widehat{\mathcal{P}}_N} \mathbb{E}^{\mathbb{Q}}\left[\max\{\ell_1(\xi), \ell_2(\xi)\}\right]$$

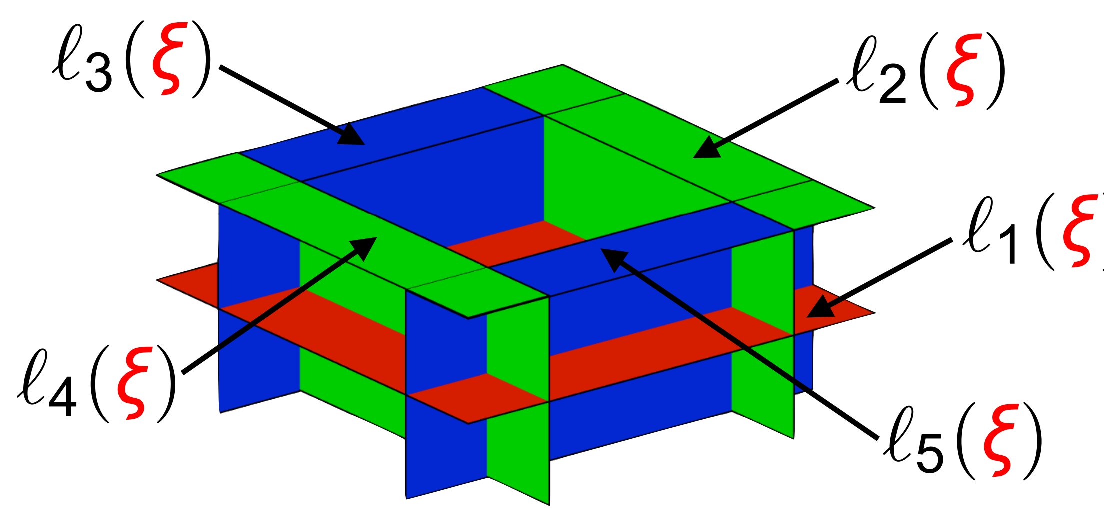




Probability of being outside of a polytope:

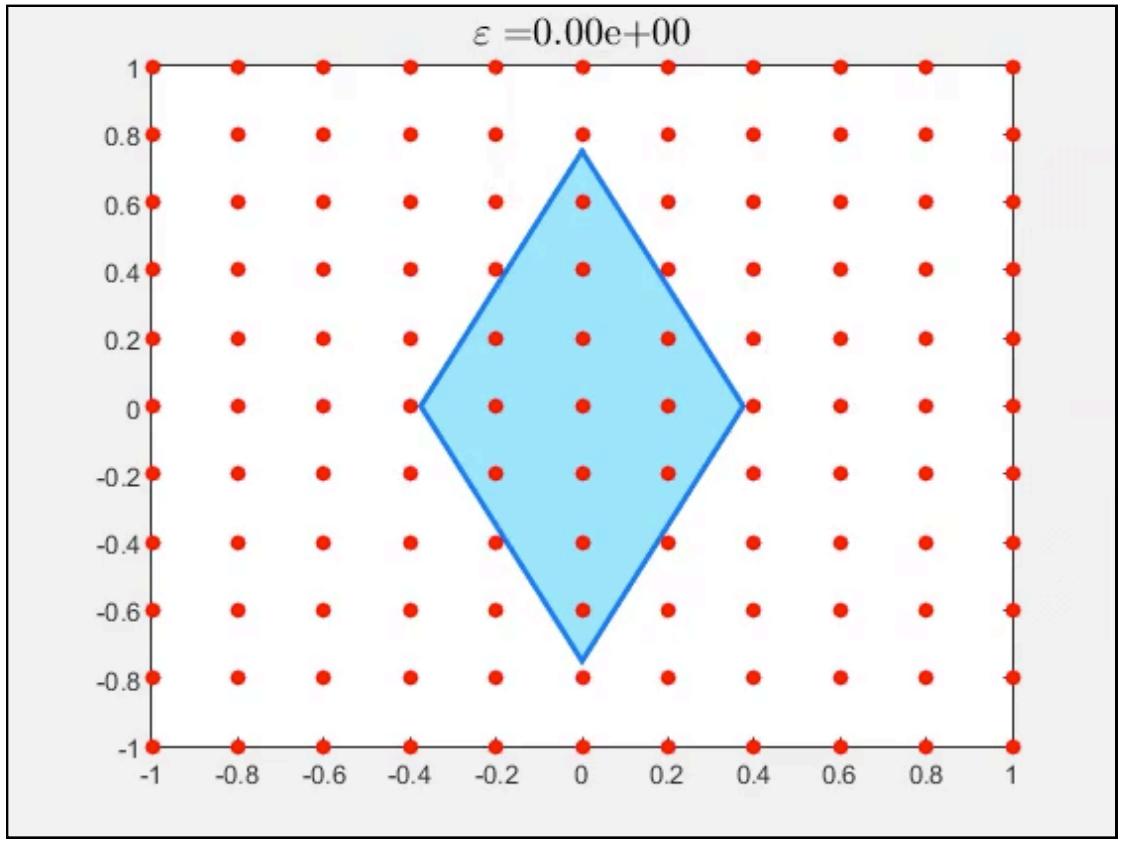
$$\sup_{\mathbb{Q}\in\widehat{\mathcal{P}}_N} \mathbb{Q}\left[\xi \notin \mathbb{A}\right] = \sup_{\mathbb{Q}\in\widehat{\mathcal{P}}_N} \mathbb{E}^{\mathbb{Q}}\left[1 - \mathbb{I}_{\mathbb{A}}(\xi)\right] = \sup_{\mathbb{Q}\in\widehat{\mathcal{P}}_N} \mathbb{E}^{\mathbb{Q}}\left[\max_{j\leq J}\{\ell_j(\xi)\}\right]$$



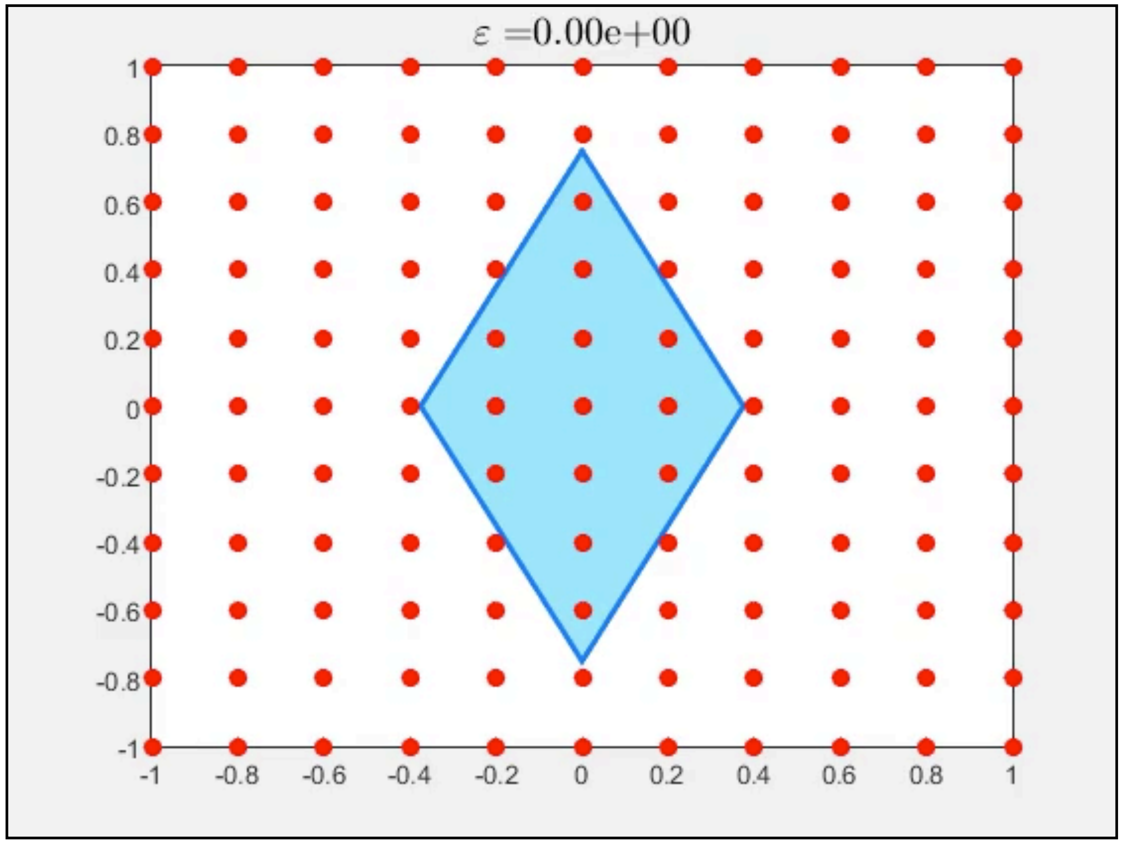


Uncertainty Quantification

Probability of being inside a polytope:



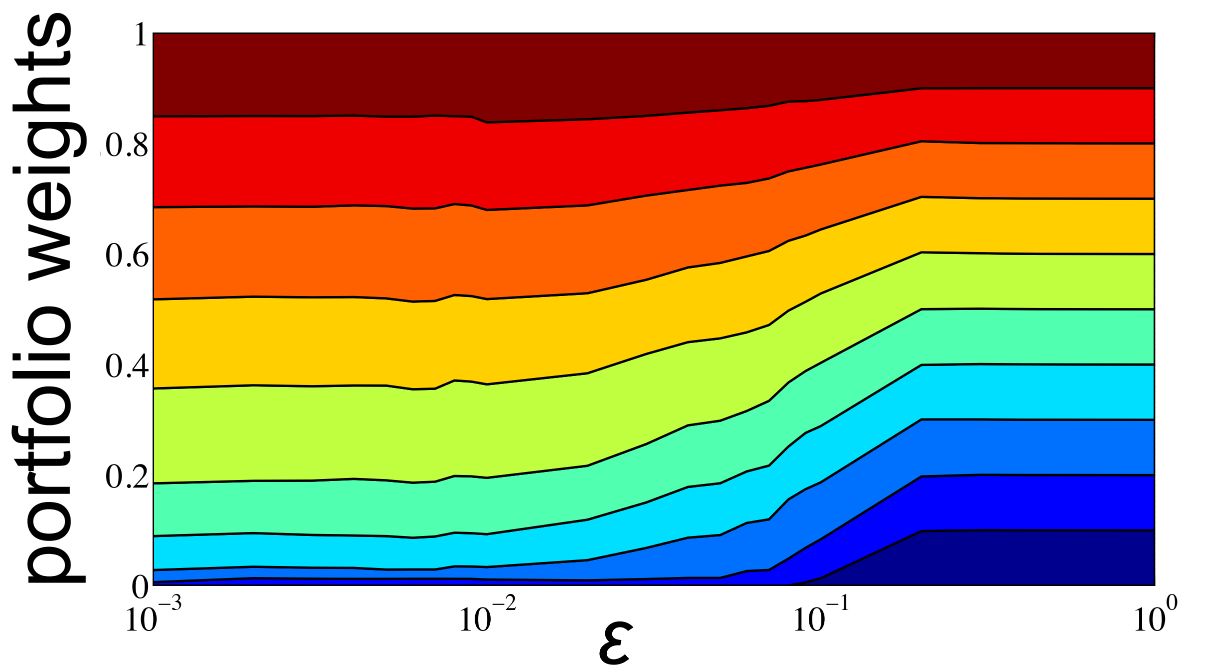
Probability of being outside of a polytope:



Application 1: Portfolio Selection

$$\min_{x \in \mathcal{X}} \left\{ \mathbb{E}^{\mathbb{P}}[-x^\top \xi] + \rho\, \mathbb{P}\text{-CVaR}_\alpha(-x^\top \xi) \right\}$$

- 10 assets
- $\rho = 10$
- $\alpha = 20\%$
- $\xi_i = \psi + \zeta_i$ where $\psi \sim \mathcal{N}(0, 2\%)$ and $\zeta_i \sim \mathcal{N}(i \times 3\%, i \times 2.5\%)$



Fact: The 1/n portfolio is hard to beat out of sample.[8)]

Possible Explanation: It is optimal for $\varepsilon \to \infty$.[9)]

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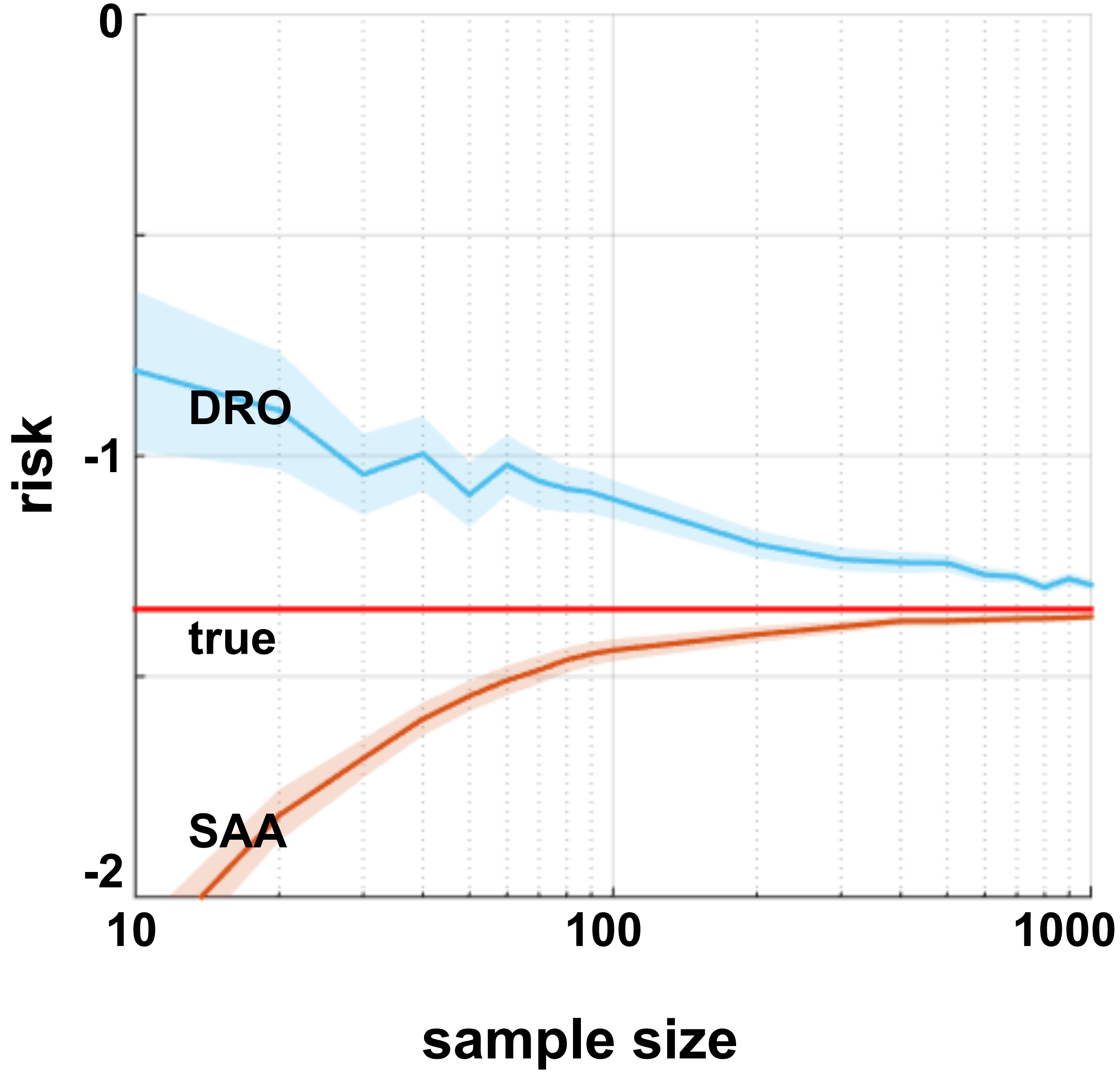
8) DeMiguel, Garlappi & Uppal, *Rev. Financ. Stud.*, 2009;
9) Pflug, Pichler & Wozabal, *J. Bank. Financ.*, 2012.

Learning Curves

what we **think** to get …



Learning Curves